INTERNATIONAL DAY OF MATHEMATICS MARCH 14

Tower of Hanoi and Tower of Stockmeyer

Preparations:

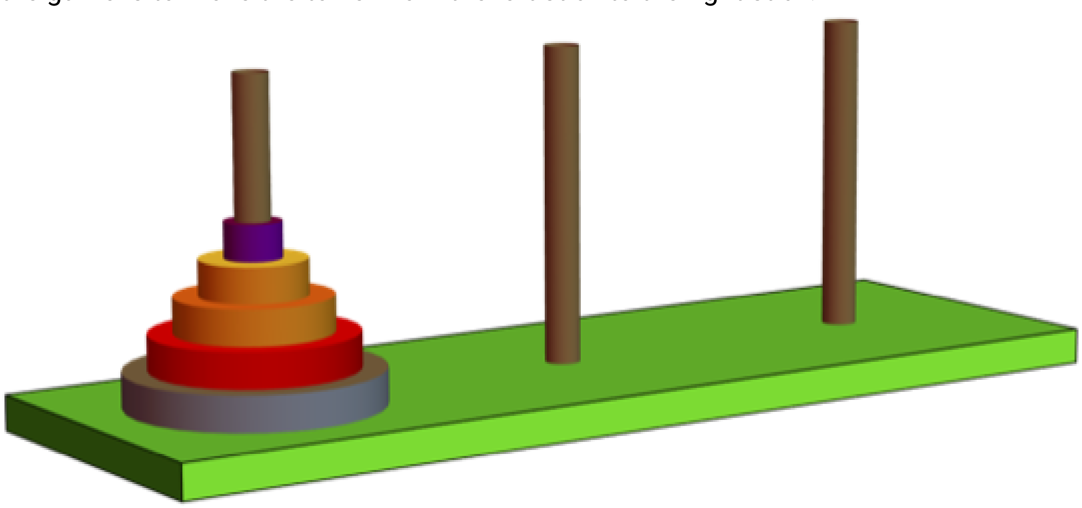
It is recommended to use a physical game as shown in the figures below.
In case there is no game available, you can build each disc by gluing several cardboard circles together and stack them in three different positions next to each other. The sticks are not necessarily required. An alternative is to use coins of different sizes.

Participants:

Ideally, provide one game per person. It is also possible to have one game per group of 2-3 persons and have the players discuss the strategy.
Age: Starting from 6 years old.
Note: This activity is also suitable for people who are blind.

Rules for playing the Tower of Hanoi:

There are *N* discs of decreasing sizes and three sticks. *N* can be any number, for example 5 as shown in the image below.
In the beginning, all discs are piled in decreasing sizes from bottom to top on one stick. The only admissible movement is to move the top disc from one stick to another stick respecting the constraint that no disc can be piled on top of a smaller disc. The objective of the game is to move the tower from the left stick to the right stick.

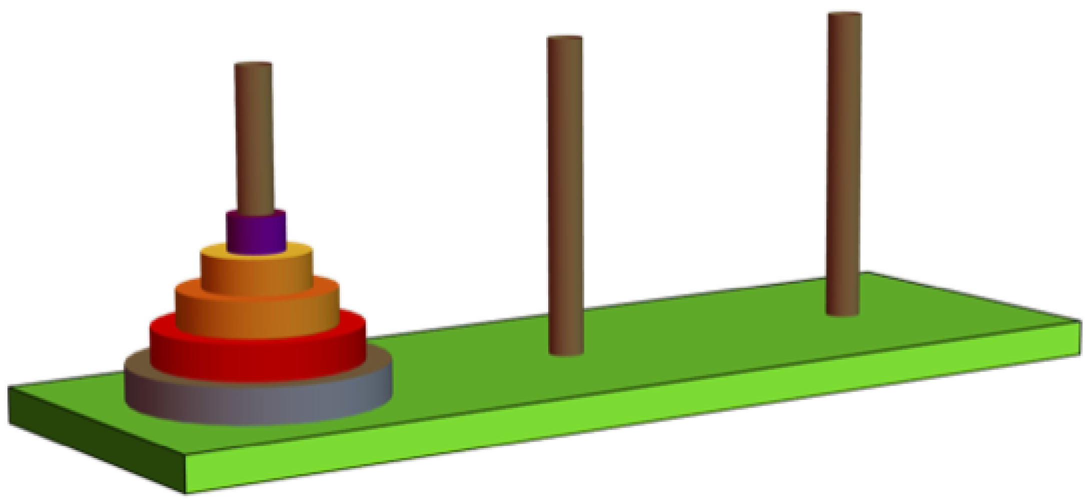


Possible tasks:

1. Start with three discs and move all the discs from the left stick to the right stick. What is the minimal number of moves?
2. Repeat the same with four discs and move all the discs from the left stick to the right stick. What is the minimal number of moves?
3. Repeat the same with five discs and move all the discs from the left stick to the right stick. What is the minimal number of moves?
4. **(This question is more difficult and requires familiarity with abstract algebra.)** Can you guess the minimal number of moves for *N* discs? Suggestion: Let a_N be the minimal number of moves for *N* discs. Compute a_N as a function of a_{N-1}.

Variant of Tower of Hanoi with moves to adjacent sticks:

There is an additional constraint that a disc must always be moved from one stick to an adjacent stick.



Possible tasks:

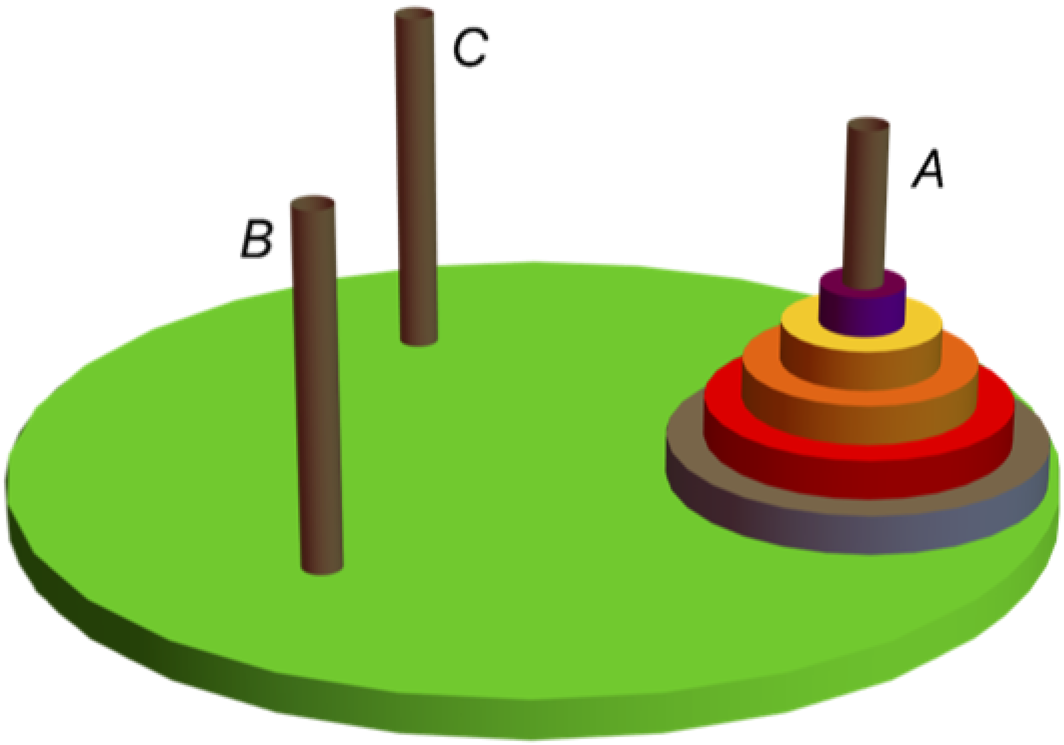
1. Play the game with three discs and move all discs from the leftmost stick to the rightmost stick. What is the minimal number of moves?
2. Repeat the same with four discs and move all the discs from the leftmost stick to the rightmost stick. What is the minimal number of moves?
3. **(This question is more difficult and requires familiarity with abstract algebra.)** Can you guess the minimal number of moves for *N* discs? Suggestion: Let b_N be the minimal number of moves for *N* discs. Compute b_N as a function of b_{N-1}.

Group discussion:

You may want to have a group discussion at this point before going to the next activities. Ask the players to explain the general procedure (and their thoughts about finding the formula).

Cyclic Tower of Hanoi:

The three sticks are now at the vertices of a triangle, and a disc can only be moved to the next stick in the clockwise direction. Let A, B and C be the three sticks in the clockwise direction.

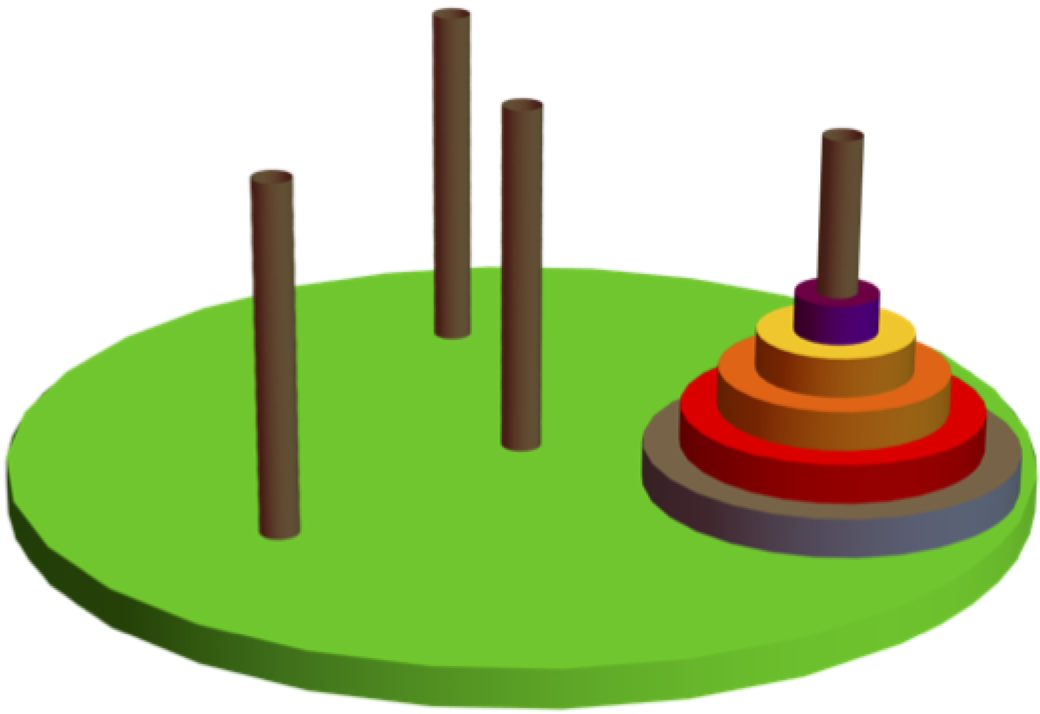


Possible tasks:

1. Play the game with three discs and move all discs from A to B. What is the minimal number of moves?
2. Repeat the same with four discs and move all discs from A to B. What is the minimal number of moves?
3. Play the game with three discs and move all discs from A to C. What is the minimal number of moves?
4. Repeat the same with four discs and move all discs from A to C. What is the minimal number of moves?

Tower of Stockmeyer:

There are *N* discs of decreasing sizes and four sticks.
Three sticks are located at the vertices of a triangle and are called *lateral sticks,* the fourth stick is at the center of the triangle and is called *central stick*. In the beginning, all discs are piled in decreasing sizes from bottom to top on one lateral stick.



The only admissible movements are to move the top disc from one lateral stick to the central stick or from the central stick to a lateral stick and - same as in the game Tower of Hanoi - to respect the constraint that no disc can be piled on top of a smaller disc.

Possible tasks:

1. Start with two discs and move all the discs from one lateral stick to another lateral stick. It is proven that the minimal number of moves is 6. Can you achieve it?
2. Repeat the same with three discs and move all the discs from one lateral stick to another lateral stick. It is proven that the minimal number of moves is 12. Can you achieve it?
3. Repeat the same with four discs and move all the discs from one lateral stick to another lateral stick. It is proven that the minimal number of moves is 20. Can you achieve it?
4. Repeat the same with five discs and move all the discs from one lateral stick to another lateral stick. It is proven that the minimal number of moves is 32. Can you achieve it?

After the game:

Discuss the different strategies. Do you want to invent new rules? For instance, increasing the number of sticks reduces the minimum number of moves for N discs. Or do you want to create a new game?

You may also check out the Numberphile video by Ayliean MacDonald, introducing the basic game (including a cardboard version without sticks), presenting a way to create music while solving the game, interesting patterns, and cool other insights.

Create and Share!
Take a video of someone playing the game very fast (best from an elevated position). You may even want to accelerate the video. Can you invent new rules? Share your creations, videos, playlists etc., using the hashtag **#idm314hanoi** and **#idm314**.

Reference:

Variations of the Four-Post Tower of Hanoi Puzzle, Paul K. Stockmeyer, Proceedings of the Twenty-fifth Southeastern International Conference on Combinatorics, Graph Theory and Computing (Boca Raton, FL, 1994). *Congr. Numer.* 102 (1994), 3–12.