# Artistic Polyhedra

**Participants:**

Ages 14 and up, depending on the activity.
No prior math knowledge is needed, but as you progress through the activities, you may need to apply mathematical concepts like polygons, regular polygons, and polyhedra, as well as use basic facts such as the sum of the angles in a triangle, the golden ratio, basic geometric constructions, or and cartesian coordinates.

**Materials:**

For the physical models, wooden skewers and rubber bands are used. Optionally, the skewers can be colored with paint. In some constructions, it may be useful to make polyhedra from string to avoid cluttering with skewers. You can also construct polyhedra with colored cardboard.

Some of the images can be projected in the classroom by the teacher. Additionally, some activities include links to animations and interactive apps online.

## 1. Regular polyhedra.

Preliminary questions:

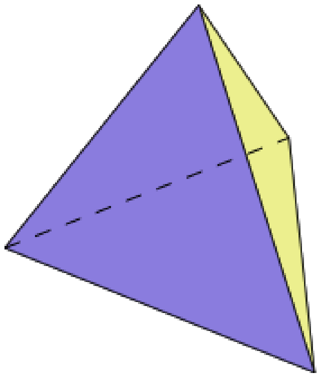
What is a *polyhedron*?

- It is a three-dimensional solid bounded by flat faces. The *faces* are flat polygons.
- The line segments where two faces meet are called *edges*.
- The points where three or more faces meet are called *vertices*.

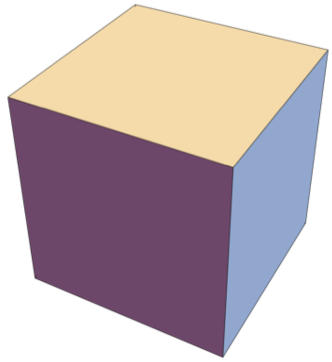
What is a regular polyhedron?

- The faces are identical regular polygons. (A polygon is regular if all its sides have the same length and all its angles are equal.)
- AND each vertex is adjacent to the same number of faces.
- AND the polyhedron is *convex*.

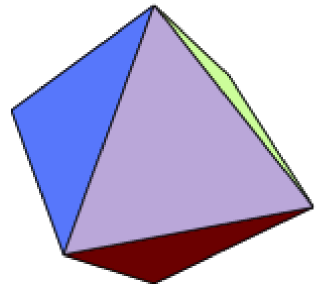
The five regular polyhedra:



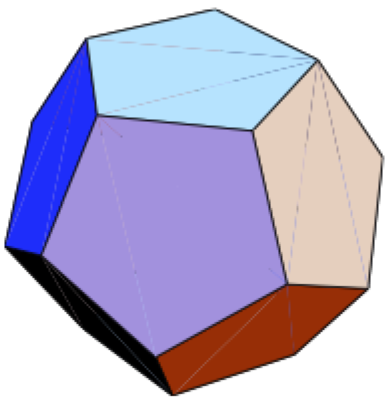
Tetrahedron



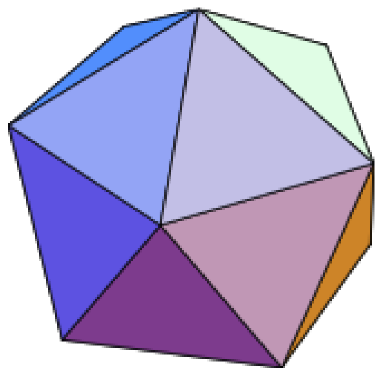
Hexahedron (Cube)



Octahedron



Dodecahedron

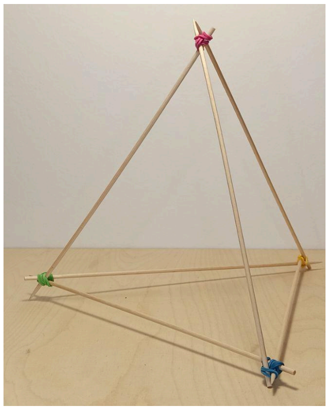
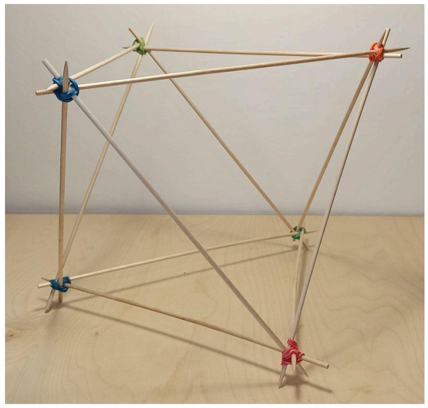
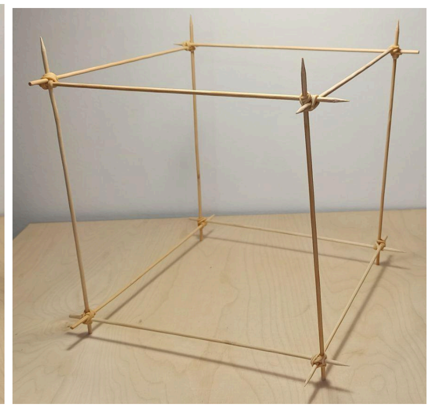
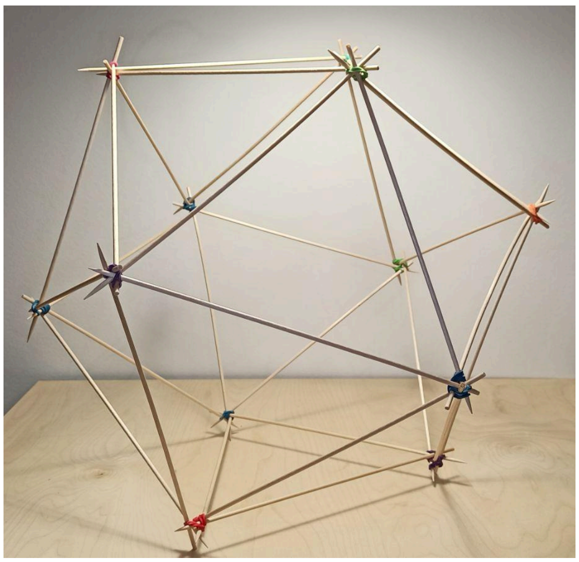
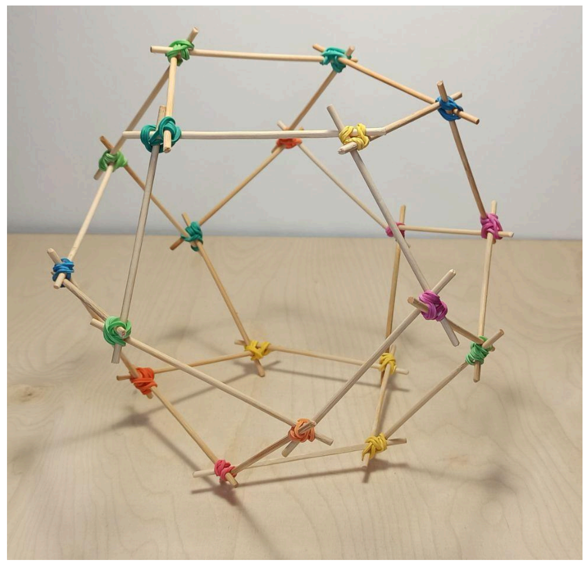


Icosahedron

These are the only five regular polyhedra, also known as Platonic solids. (click on the links to manipulate the polyhedra)

From the pictures and the number and type of faces, deduce the number of edges and vertices (how many skewers and rubber bands you will need) and build the five regular polyhedra.

| Regular polyhedron | Type of face | Number of faces | Number of vertices | Number of edges |
|---|---|---|---|---|
| Icosahedron | triangle | 20 | 12 | 30 |
| Octahedron | triangle | 8 | 6 | 12 |
| Tetrahedron | triangle | 4 | 4 | 6 |
| Cube | square | 6 | 8 | 12 |
| Dodecahedron | pentagon | 12 | 20 | 30 |

Observe Euler's relation:

$$F + V = E + 2$$

where F, V, and E are the number of faces, vertices, and edges, respectively.
Also, observe that there is a close relationship between:

- The octahedron and the cube.
- The icosahedron and the dodecahedron.
- The tetrahedron and itself.

(Compare the numbers of vertices, edges, and faces.)

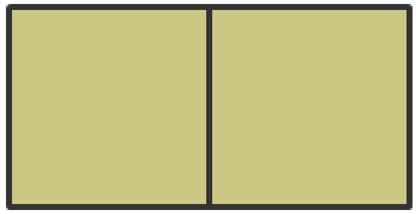
They are called *dual*: The vertices of one correspond to the faces of the other, and vice versa. You can obtain them by placing the vertices of one at the center of the faces of the other. The edges correspond one-to-one, but they are rotated 90º in the dual polyhedron.

## Only five regular polyhedra

Let's prove that there are only five regular polyhedra. You can use paper or cardboard and adhesive tape to build the figures in the argument.

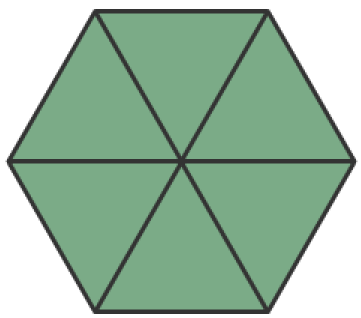
We know that in a regular polyhedron, all the faces are equal regular polygons, and let *n* be the number of faces that coincide in a vertex.

Step 1: If $n = 2$, explain why it is not possible to build any kind of polyhedron.



Step 2: We already know that $n \geq 3$. Let's try to build all possible regular polyhedra, starting by using equilateral triangles.

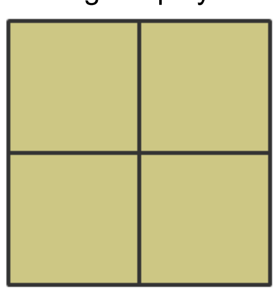
- If $n = 3$, is it possible to build a regular polyhedron? If possible, build it by joining 3 triangles in each vertex.
- If $n = 4$, is it possible to build a regular polyhedron? If possible, build it by joining 4 triangles in each vertex.
- If $n = 5$, is it possible to build a regular polyhedron? If possible, build it by joining 5 triangles in each vertex.
- If $n = 6$, it's not possible to build a regular polyhedron. Why not?



- If $n > 6$, it's not possible to build a regular polyhedron. Why not?

Step 3. Let's move on to squares.

- If $n = 3$, is it possible to build a regular polyhedron? If possible, build it by joining 3 squares in each vertex.
- If $n = 4$, it's not possible to build a regular polyhedron. Why not?



- If $n > 4$, it's not possible to build a regular polyhedron. Why not?

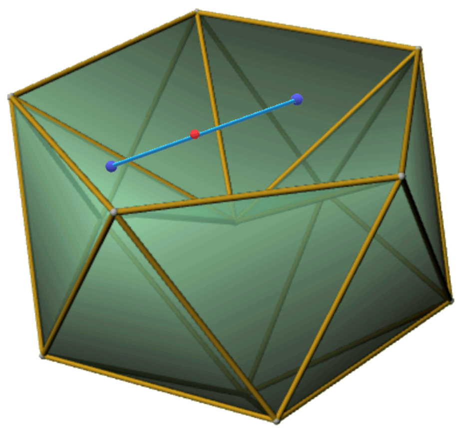
Step 4. Let's move on to regular pentagons.

- If $n = 3$, is it possible to build a regular polyhedron? If possible, build it by joining 3 pentagons in each vertex.
- If $n \geq 4$, it's not possible to build a regular polyhedron. Why not?

Step 5. Finally, let's use hexagons. Explain why it wouldn't be possible to build a regular polyhedron using regular hexagons.
Explain why it wouldn't be possible to build a regular polyhedron using regular heptagons (7 sides), regular octagons (8 sides), etc.

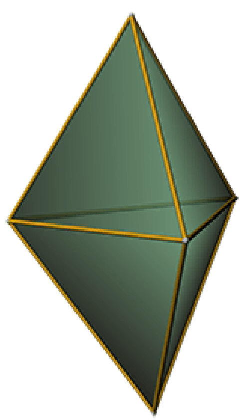
## Non-regular polyhedra

A polyhedron is convex if, for each couple of points on the faces, the segment joining the points is entirely contained in the polyhedron.



Example of a non-convex polyhedron (click here to manipulate the polyhedron)

In a regular polyhedron, all the vertices are identical (they have the same number of faces coinciding.



An example of a polyhedron with all its faces regular polygons, but not a regular polyhedron (at some vertices coincide three faces and on others four). (Click here to manipulate the polyhedron).

The definition of regular polyhedra forbids these cases.

- You can play on this app [Mathina] to distinguish between convex and non-convex polyhedra.
- You can truncate, stellate, and make other modifications to polyhedra to obtain new solids with this app [IMAGINARY GitHub].

# Euler's relation

We have already observed a relationship between the number of faces, edges and vertices in the Platonic solids. This relation applies to many more non-regular polyhedra:

**Euler's relation**. For any polyhedron equivalent to the sphere, the following relation holds

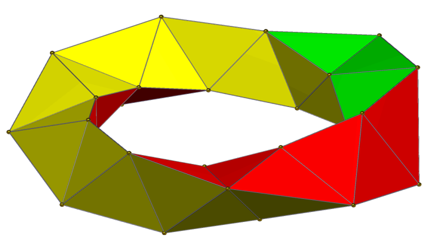
$$V - E + F = 2,$$

where $V$ is the number of vertices, $E$ is the number of edges, and $F$ is the number of faces.

More generally, Euler's characteristic, denoted by the Greek letter $\chi$, is defined as

$$\chi = V - E + F.$$

The theorem states that $\chi = 2$ for most of the polyhedra you may think of. Exceptions, that is, polyhedra not equivalent to a sphere, include, for instance, a polyhedral torus, or the Kepler-Poinsot polyhedra (when thought as having intersecting faces) because they wrap around the sphere more than once. All convex polyhedra have $\chi = 2$.

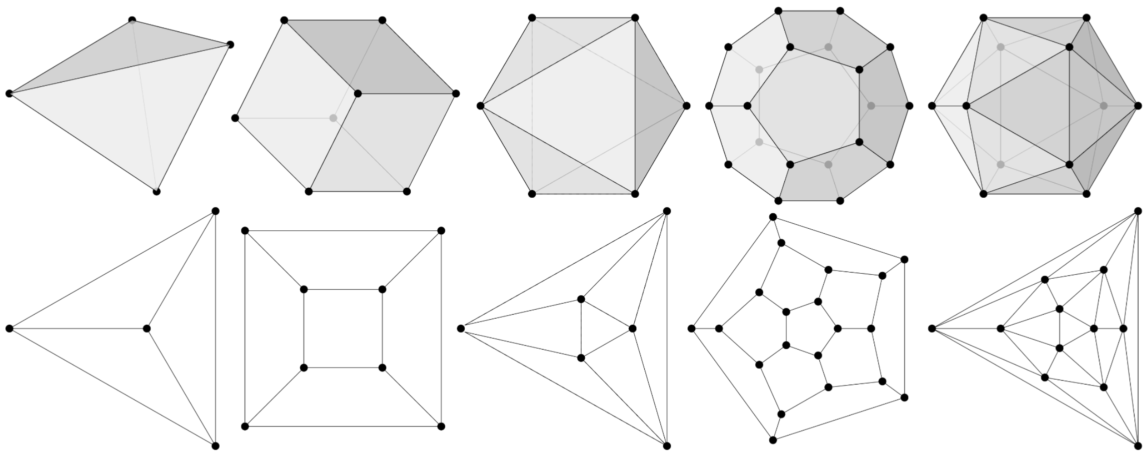


A non-regular, non-convex polyhedron with the shape of a torus, with $\chi = 0$.

Let us prove Euler's relation.

Step 1. Since the relation does not involve angles, lengths, or other metric measurements, we can deform the skeleton (i.e, the edges and vertices) as a flexible graph and flatten it in a plane. Prove that one can draw the skeleton of a convex polyhedron over a flat plane, without having the edges intersecting, yielding a *planar* graph. A graph is called *planar* when its edges do not cross (they touch only at their endpoints, which are vertices).

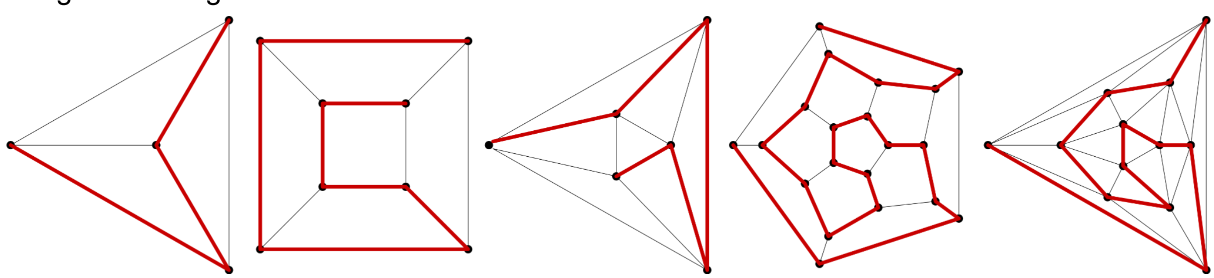
For instance, we can do it with the Platonic solids:



Note that each graph has the same number of vertices, edges, and faces as the original polyhedron. The “faces” are the enclosed regions, together with the exterior unbounded region.

We can therefore prove Euler’s relation for planar graphs.

Step 2. Show that on each planar graph, we can find a path that connects all the $V$ vertices, using $V - 1$ edges.



Step 3. Assume that we remove all edges except those on the path connecting the vertices. Prove that for this graph, we have $\chi = 2$.

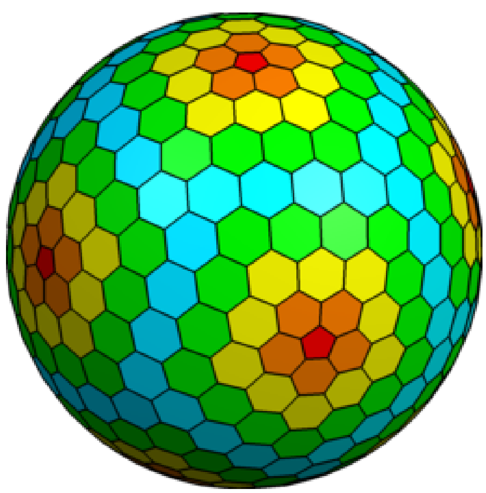
Step 4. Add back the edges removed in the previous step, one by one. Prove that by adding each edge, the Euler characteristic remains $\chi = 2$, until we recover the graph of our original polyhedron.

## Extra challenges

- Shapes with pentagons and hexagons have been used for architectural design, for instance, the *Montreal Biosphere* designed by Buckminster Fuller in 1967. In the Montreal biosphere, the faces are triangles assembled by 5 or 6 at each vertex. Using Euler’s formula, show that (if the sphere was complete) there are always exactly twelve vertices attached to five triangles.

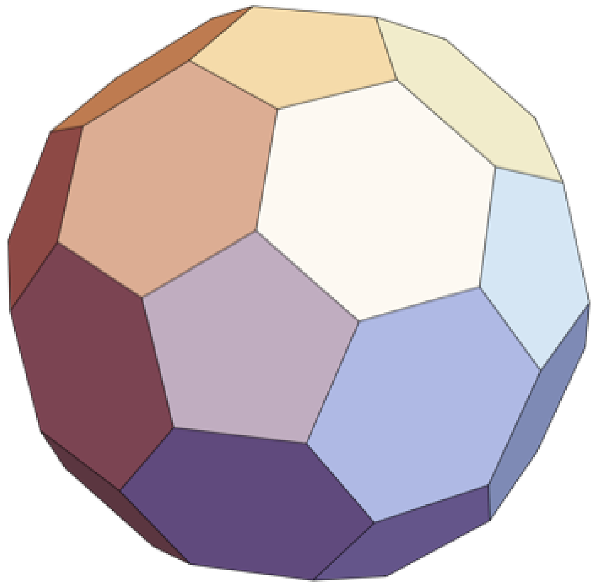
The Montreal biosphere
Image: Cédric THÉVENET, via Wikimedia Commons, CC BY-SA 3.0

- *Goldberg polyhedra* are polyhedra whose faces are all pentagons and hexagons and have the same symmetries as the icosahedron. Using Euler's formula, show that any polyhedron, whose faces are pentagons and hexagons, has exactly twelve pentagons. (Note that the icosahedron has exactly 12 vertices and five triangles are attached to each vertex.)



A Goldberg polyhedron
Image: Tomruen, via Wikimedia Commons, CC BY-SA 3.0

Another Goldberg polyhedron is the truncated icosahedron, the soccer ball we know well.



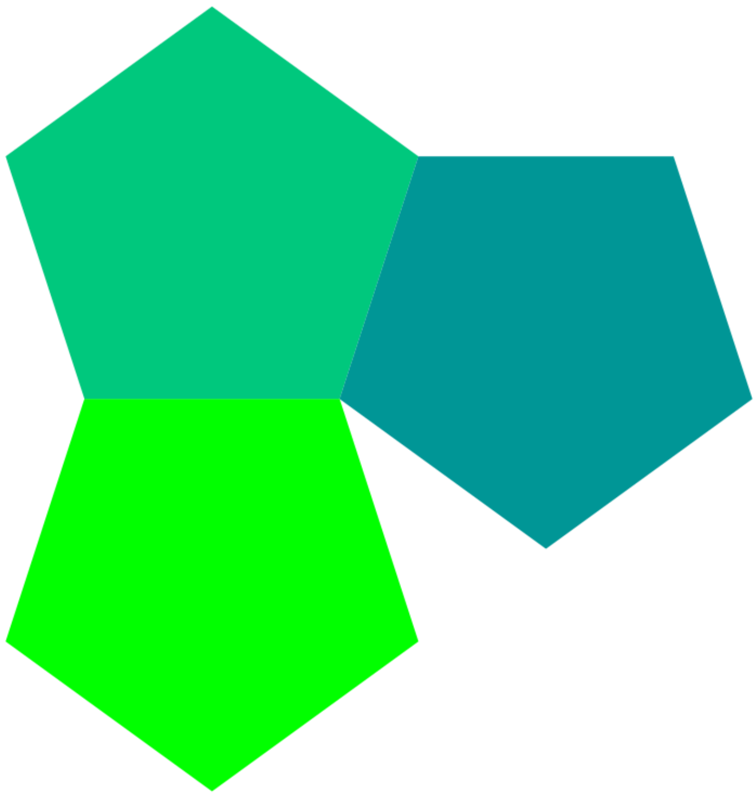
The truncated icosahedron

# Descartes’ theorem

When you build a polyhedron, you assemble faces at each vertex. You can count the sum of the angles of the faces adjacent to a vertex. This sum is

- Smaller than 360º if the polyhedron is convex near that vertex;
- Equal to 360º if the polyhedron is flat near that vertex;
- Larger than 360º if the polyhedron is concave near that vertex.

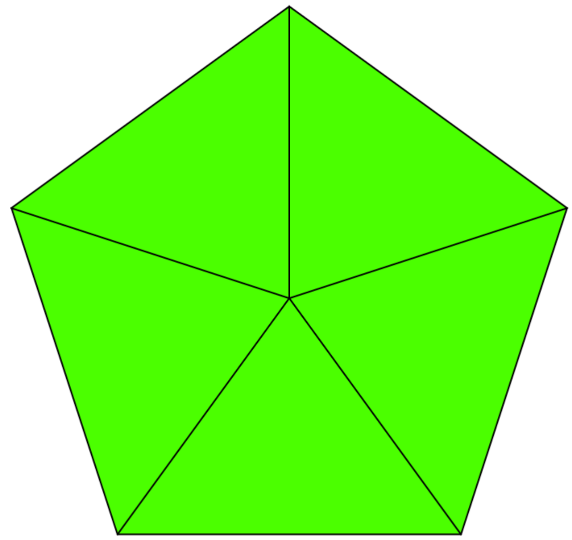
The difference between 360º and the sum of the angles at one vertex is called the *defect at that vertex.* (Note that the defect is negative in the third case.)



The defect at a vertex of a dodecahedron is 360º − 3x108º = 36º.

**Descartes’ theorem:** For a polyhedron, the sum of the defects at each vertex, also called the *total defect*, equals 720º.

a) Check Descartes’ theorem on the regular polyhedra, or other polyhedra, such as the truncated icosahedron.
b) Prove Descartes’ theorem using Euler’s formula:
    i) First, you can reduce the problem to that of a polyhedron with triangular faces. For that, it suffices to take an inner point inside each polygonal face and join it to all vertices of the face, thus dividing the face into triangles.



If a face has *n* edges, the process adds one vertex (the inner point), *n* edges joining the inner point to the vertices, and a face is replaced by *n* triangular faces. Hence Euler's formula ($V + F = E + 2$) remains valid.

ii) Prove Descartes' theorem for a polyhedron with triangular faces. Let $D$ be the sum of the defects, $V$ be the number of vertices, $E$ be the number of edges, and $F$ be the number of faces. Then:

$$E = 3/2\ F.$$

since each Face has three edges, and each edge gets counted twice.

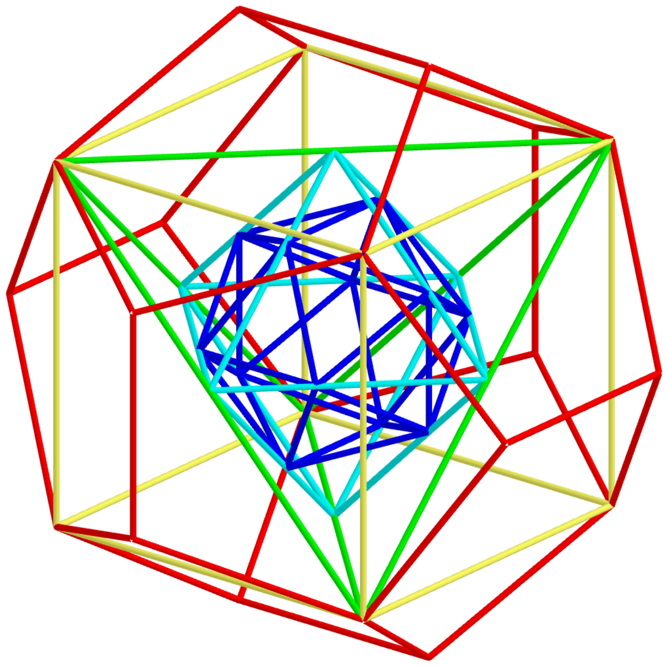
$$D = 360^\circ\ V - 180^\circ\ F.$$

Thus,

$$D = 180^\circ(2V - F) = 180^\circ(2V + 2F - 3F) = 360^\circ(V + F - 3/2F)$$
$$= 360^\circ(V + F - E) = 360^\circ \times 2 = 720^\circ.$$

# Building the omnipolyhedron

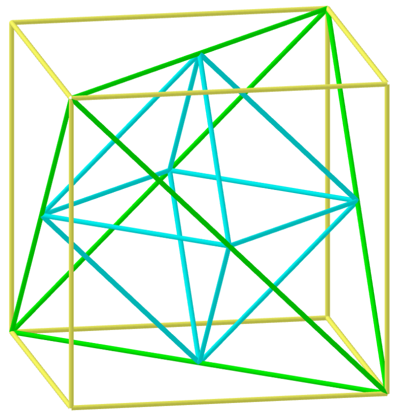
An arrangement of the five regular polyhedra inscribed into each other that displays some of their relations and properties is sometimes called an *omnipolyhedron* (it is not a single polyhedron, but an arrangement of polyhedra).

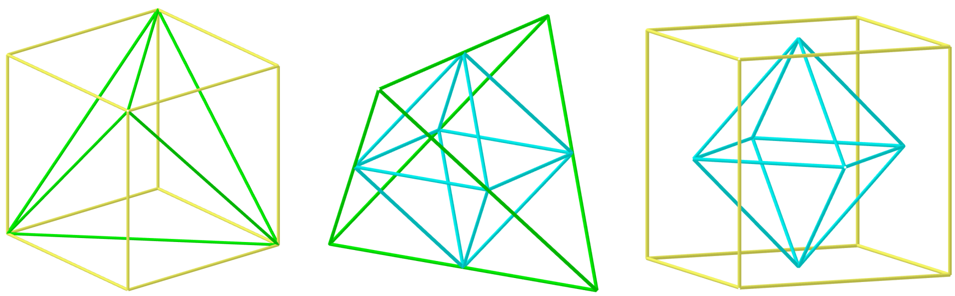


We propose three partial constructions of inscribed polyhedra. You can make any of these, or combine the three to build an omnipolyhedron. You can use wooden skewers and rubber bands. The icosahedron can also be made of string once the octahedron is built. You can also buy a kit from Zometool.

## The cube, the tetrahedron, and the octahedron.

Build a cube. Add one diagonal on each of the six faces, building a tetrahedron: for that purpose, you need to choose diagonals, all of whose endpoints are on four vertices of the cube, and three diagonals are attached to each of these four vertices (there are two ways to do it.) Join the midpoints of the tetrahedron's edges, building an octahedron.



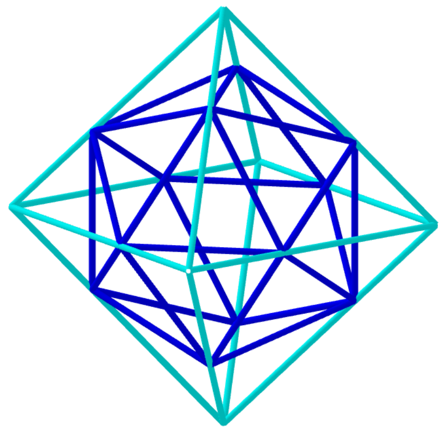


Sketch this construction on paper, build it with sticks, and prove mathematically that it works. Calculate the lengths of the tetrahedron and octahedron’s edges.

Note that the edges of the octahedron are at the center of the faces of the cube, so we can see that the octahedron is the cube's dual. You obtain the dual tetrahedron if you choose the other diagonals of the cube’s faces.

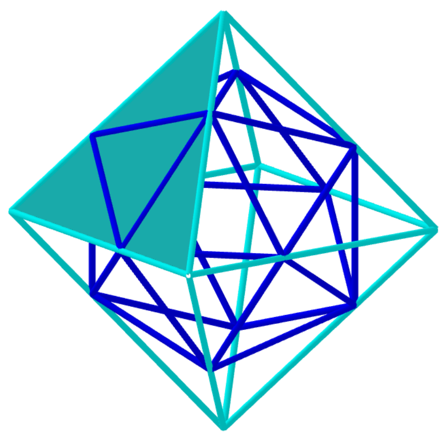
## The icosahedron inscribed into the octahedron

A regular icosahedron can be inscribed into a regular octahedron so that each of the twelve vertices of the icosahedron lies on one of the twelve edges of the octahedron, and these vertices split the edges in a golden ratio ($\phi = \frac{1+\sqrt{5}}{2} \simeq 1.618...$ is the *golden ratio*, given by the equation $\frac{\phi}{1} = \frac{\phi+1}{\phi}$, or equivalently $\phi^2 = \phi + 1$),

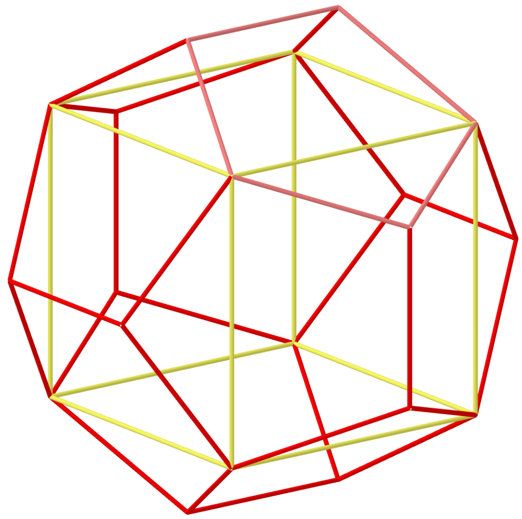


Sketch this construction on paper, build it with sticks, and prove mathematically that it works.

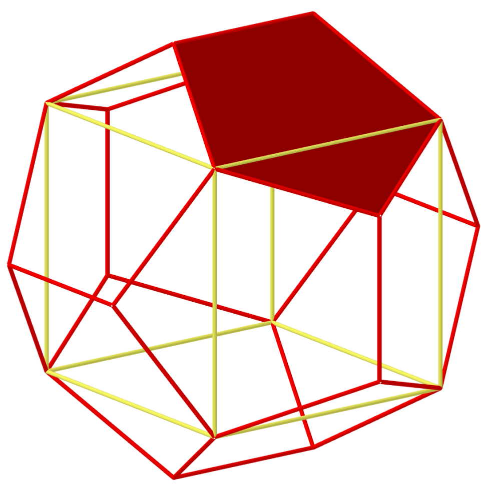
Hint: Each face of the octahedron contains one triangular face of the icosahedron. Compute the length of its side. The rest of the icosahedron's edges lie on the octahedron's interior. Prove that their length is the same as the previously built edges.



## The dodecahedron circumscribed to the cube



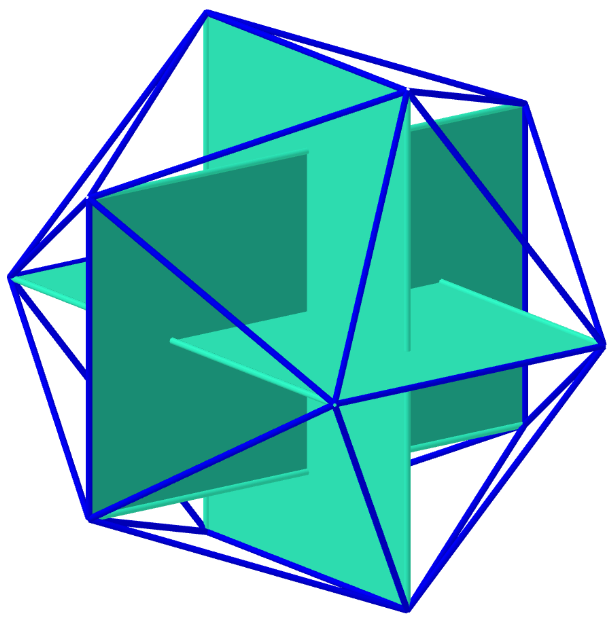
Take a cube. On top of each square face, build a “roof” made of two triangles on opposite sides, and two trapezoids on opposite sides, using five sticks (first build the two triangles by adding two sticks to each of two opposite sides, then join the free vertices with the remaining stick). Make the roofs on each of the cube’s faces such that a trapezoid meets a triangle. If the length of the new edges is exactly $1/\phi$ of the cube’s edge, then the trapezoid and the triangle paired around a cube’s edge will align perfectly to form a regular pentagon, and the global result will be a regular dodecahedron.



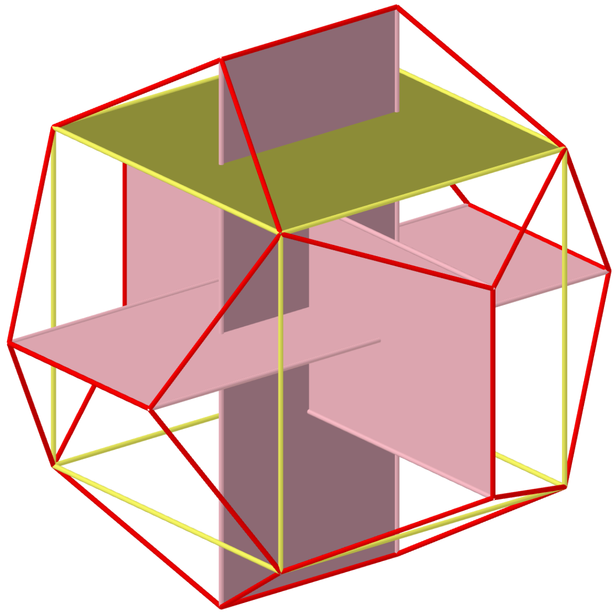
Sketch this construction on paper, build it with sticks, and prove mathematically that it works.

## Extra challenges

- Take three golden rectangles (in a *golden rectangle* the length is ϕ times the width.) Cut a slit on the center of each of them (the size is the width of the rectangles), and assemble them such that each rectangle is perpendicular to the other two. The twelve vertices are arranged as in an icosahedron. Prove it and build it.
  Note: Most credit cards and business cards are golden rectangles. You can use these cards and some string to make the edges.



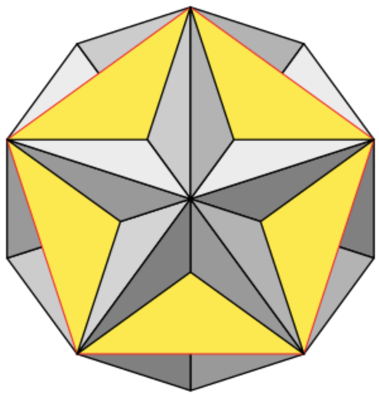
- Take three rectangles whose length is $\phi^2$ times the width  Cut a slit on the center of each of them, and assemble them such that each rectangle is perpendicular to the other two. Insert this construction into the center of a cube of side ϕ, keeping the rectangles parallel to the faces. Then, the twelve vertices of the three rectangles, plus the eight vertices of the cube, form the set of vertices of a regular dodecahedron.



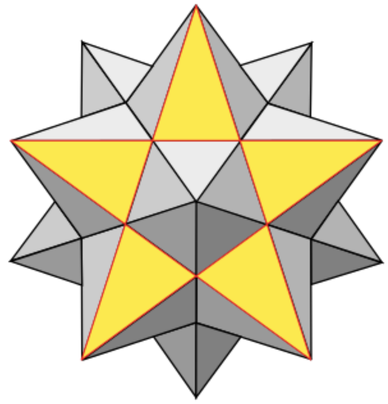
- Build a model in GeoGebra by calculating the cartesian coordinates of all the vertices.
  Solution:
  - Cube: $(\pm 1, \pm 1, \pm 1)$
  - Tetrahedron: the subset of vertices of the cube, such that there is an even number of minus signs.
  - Octahedron: $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$, $(0, 0, \pm 1)$
  - Icosahedron: $\left(\pm \frac{1}{\phi}, 0, \pm \left(1 - \frac{1}{\phi}\right)\right)$, and their cyclic permutations, namely $\left(0, \pm \left(1 - \frac{1}{\phi}\right), \pm \frac{1}{\phi}\right)$ and $\left(\pm \left(1 - \frac{1}{\phi}\right), \pm \frac{1}{\phi}, 0\right)$.
  - Dodecahedron: These of the cube, together with $\left(\pm \frac{1}{\phi}, 0, \pm \phi\right)$, and their cyclic permutations, namely $\left(0, \pm \phi, \pm \frac{1}{\phi}\right)$ and $\left(\pm \phi, \pm \frac{1}{\phi}, 0\right)$

| Regular polyhedron | Number of faces | Number of vertices | Number of edges | Edge length |
|---|---|---|---|---|
| Icosahedron | 20 | 12 | 30 | $\frac{1}{\phi^2}$ |
| Octahedron | 8 | 6 | 12 | $\frac{\sqrt{2}}{2}$ |
| Tetrahedron | 4 | 4 | 6 | $\sqrt{2}$ |
| Cube | 6 | 8 | 12 | 1 |
| Dodecahedron | 12 | 20 | 30 | $\frac{1}{\phi}$ |

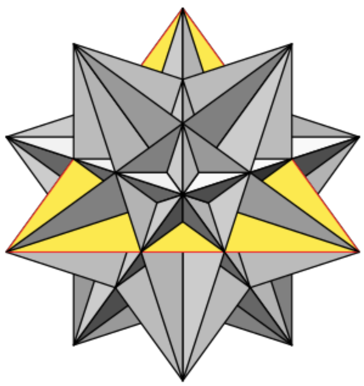
# The Kepler-Poinsot polyhedra



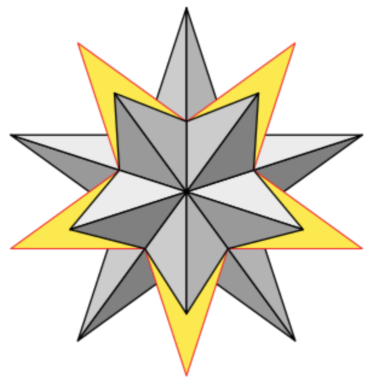
Great dodecahedron



Small stellated dodecahedron



Great icosahedron

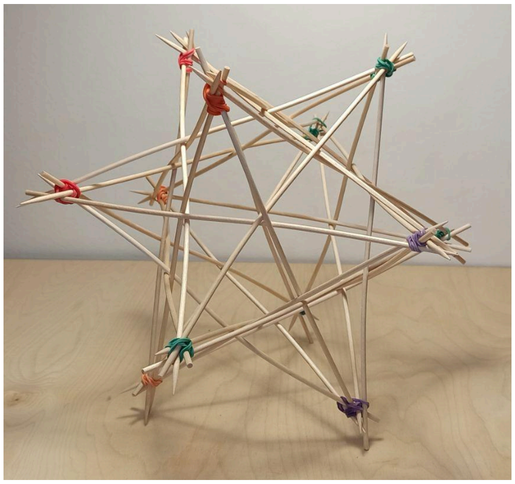


Great stellated dodecahedron

There are four Kepler-Poinsot polyhedra, and they are all non-convex. They seem to have lots of faces, edges and vertices. For instance, to build a *small stellated dodecahedron* out of cardboard, you need to cut and glue 60 faces. This polyhedron has 90 edges and 32 vertices. Hence, Euler's relation is again true.

But mathematicians have a lot of imagination and creativity. Indeed, look again at the small stellated dodecahedron. And look at the five yellow triangles. They all lie in the same plane. If we are to complete the missing middle part (which has the shape of a pentagon, what we have is a regular five-pointed star, called a *pentagram*. Hence, mathematicians can decide to view the middle part of the star, not as missing, but as lying inside the polyhedron. The same is true for all other small faces. By groups of five, they belong to other pentagrams, whose middle part lies inside the polyhedron. Since we started with 60 faces, this will result in twelve pentagrams. Hence we can consider this small stellated dodecahedron as having twelve (hence the name) *generalized faces* in the shape of pentagrams, that are allowed to intersect! Mathematicians decide to enlarge the definition of polyhedron, so as to allow such a construction, You are now invited to analyse in a similar fashion the three other Kepler-Poinsot polyhedra. The two *stellated dodecahedra* have twelve faces, which have the shape of pentagrams. The *great dodecahedron* has twelve intersecting pentagonal faces. The g*reat icosahedron* has twenty intersecting triangular faces.

You can try to build some of these polyhedra out of skewers and/or cardboard.



# Other resources

- Read a fantasy story about poyhedra online, with interactive apps and films.
- The Portuguese association Atractor has many animations and images of polyhedra (and their nets, properties, etc) on their website.
- Zometool. Kepler's obsession. https://www.zometool.com/products/keplers-obsession.html

**Create and Share!**
Share the participants' findings using the hashtags **#idm314polyhedra** and **#idm314**.