INTERNATIONAL DAY OF MATHEMATICS
MARCH 14

The Mathematics of Eclipses

A lunar eclipse will occur on [March 13-14, 2025](), precisely on IDM 2025. This will be a total lunar eclipse visible over a large part of the Americas. Six months later, on [September 7-8, 2025](), a similar lunar eclipse will occur, visible totally in most of Asia, east Africa, and Australia. This may be a good opportunity to discuss the mathematics of eclipses.

Participants:

Ages 13 and up
No prior mathematical knowledge is needed, but participants should have a basic understanding of the movements of the Earth and the Moon.

Activity:

Making a physical model of the Sun, the Earth, and the Moon.
Discussing the mathematics behind the phenomenon of eclipses.

Materials:

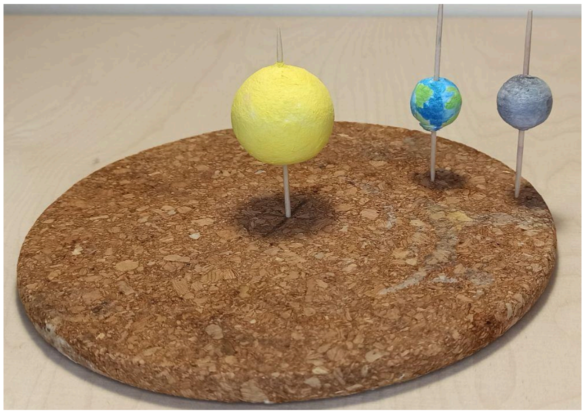
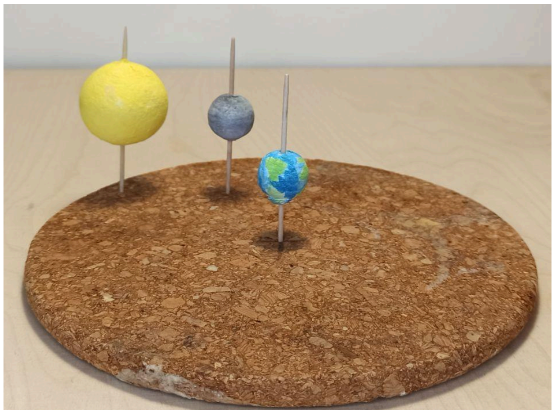
- Three balls made of compressed cotton, styrofoam, wood, etc. The balls need to have a pass-through hole. Two or three sizes is better.
- Toothpicks that can be stuck through the balls.
- Watercolor or another type of paint that is suitable for coloring the balls.
- Cork or cardboard surface to pin the toothpicks.

Preparation:

- Paint one big ball yellow (the Sun), a medium ball green and blue (Earth) and a small ball grey (Moon).

1. Planar model

Task: Pin the three balls on cardboard as they are in reality and describe how they move.

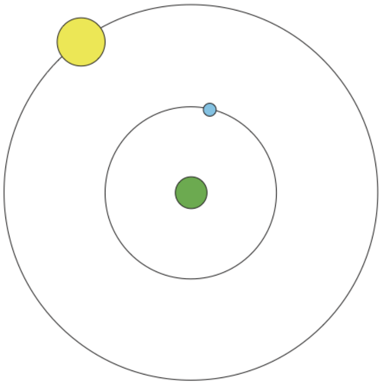



The two main relative movements of the Sun, Earth and Moon are:

- The Earth rotates around the Sun.
- The Moon rotates around the Earth.

You can put the Sun in the center, the Earth at some distance, and the Moon close to the Earth. The sizes and distances will certainly not be to scale, but only rough estimates are needed.

Another option is to put the Earth in the center, and the Sun and Moon moving around. This is also a valid model. This illustrates a very important idea: the “center” is just a matter of which reference you take (there is no absolute center of the universe).



The first case is a Heliocentric model (Helios = Sun), and the second one is a Geocentric model (Geo = Earth). Both views are equivalent, but sometimes one point of view is better for understanding of the phenomenon or for computations. Mathematicians do use changes of point of view to these purposes.

Not all configurations are valid, for example, putting the Sun between the Earth and the Moon is wrong.

Assuming the North Pole of the Earth is upwards, the Earth should move anticlockwise around the Sun (or the Sun around the Earth). The Moon should also move counterclockwise around the Earth. The Earth also rotates on its axis counterclockwise. Note that we are placing all bodies on the same plane, which is not how they are in reality.

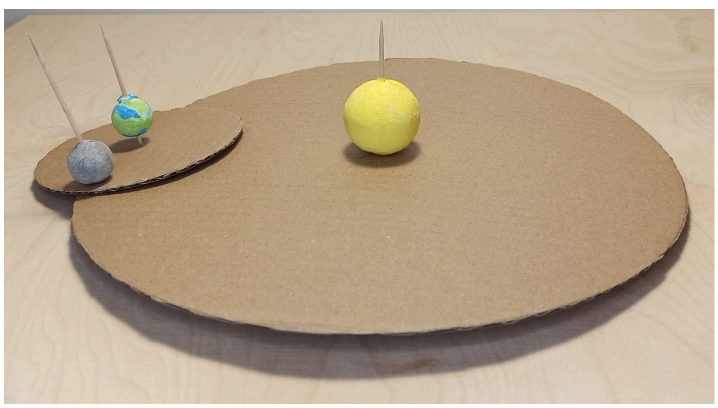
2. Moon phases and eclipses.

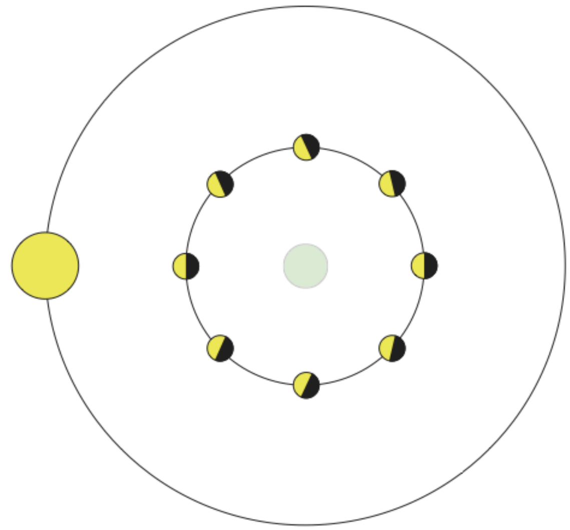
Task: Write down a definition of the following astronomical events, together with building a model with the balls in the cork:

a. New Moon
b. First quarter
c. Full Moon
d. Last quarter
e. Solar eclipse (total or partial)
f. Lunar eclipse (total or partial)

Discuss: A basic answer would be:

- The Sun can only illuminate half of the Moon. The phases of the Moon happen because only part of the illuminated half of the Moon is facing the Earth.
- Eclipses happen when the three celestial bodies are somehow aligned, and one hides the other.

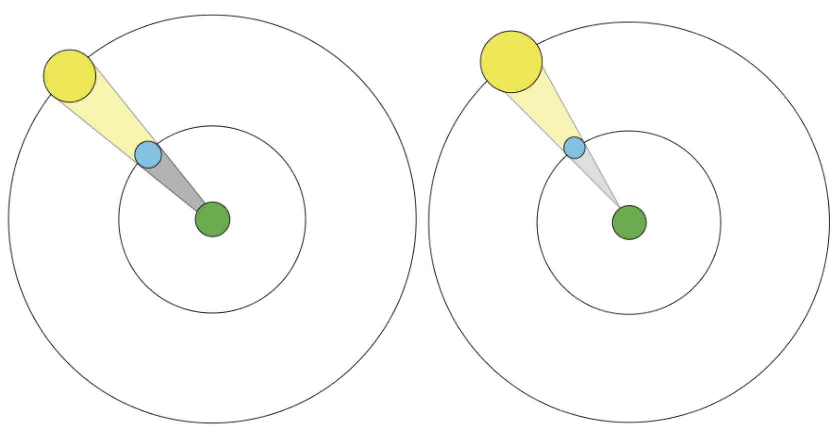




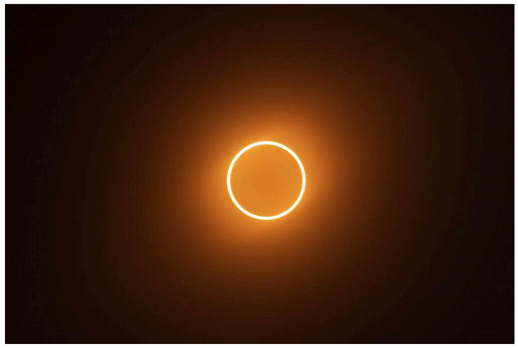
Phases of the moon

The answers we are looking for are:

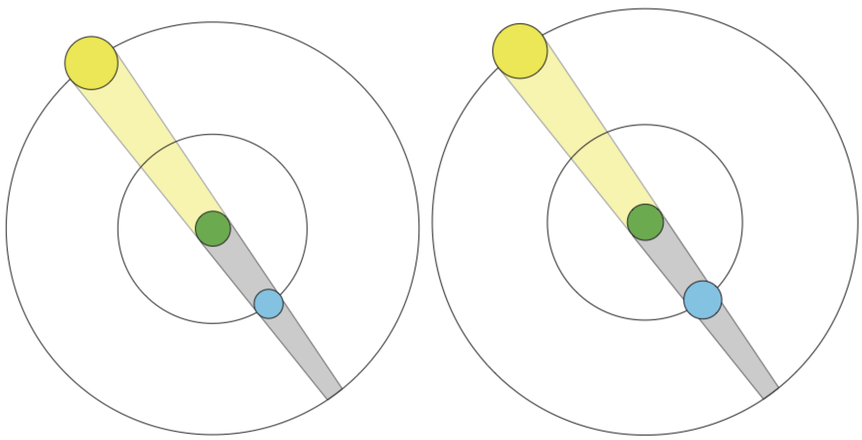
a. *New Moon.* The Moon is between the Earth and the Sun (but not aligned). The side of the Moon that is illuminated is facing away from us, and we are facing the side that is not illuminated by the Sun (it is in the shadow). In the parts of the Earth where it's daytime, the Moon is in the sky but it is not visible (since it is not illuminated), from the parts of the Earth where it's nighttime, the Moon is not in the sky (it is hidden behind the horizon).
b. *First quarter.* The Sun, the Earth, and the Moon form a triangle roughly right-angled at the Earth. The Moon is visible before and after the sunset.
c. *Full Moon.* The Earth is between the Sun and the Moon (but not aligned). We can see almost the full Moon hemisphere illuminated by the Sun for most of the night.
d. *Last quarter.* The Sun, the Earth, and the Moon form a triangle roughly right-angled at the Earth. The Moon is visible before and after the sunrise.
e. *Solar eclipse* (total, annular, or partial). The Earth is in the shadow cast by the Moon, so the Sun disappears momentarily behind the Moon.
f. *Lunar eclipse* (total or partial). The Moon is in the shadow cast by the Earth, so the Moon is momentarily obscured.



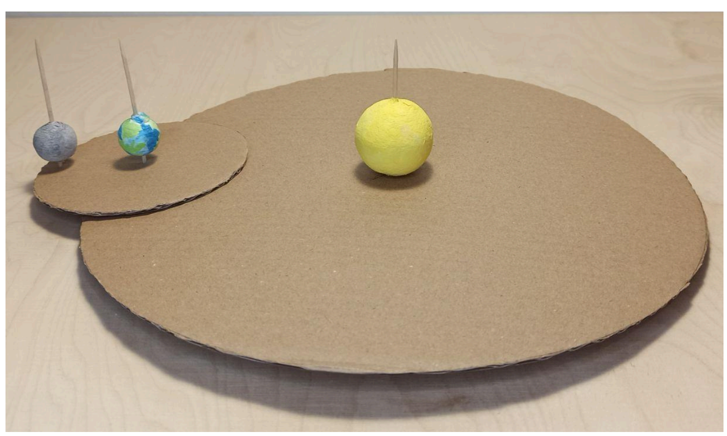
A total solar eclipse, and an annular solar eclipse



Photograph of an annular solar eclipse
Source: Dpickd1 via Wikimedia Commons, CC-BY 4.0 ,



Total lunar eclipse (it happens), and Annular lunar eclipse (it can never happen)



3. Rotation periods

How long does it take for each body to complete one turn?

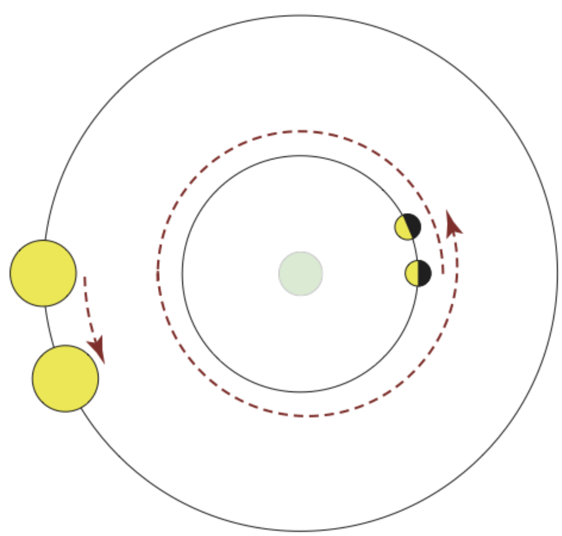
A first answer from measurement would be:

- The period of the rotation of the Earth around its axis is one day;
- The period of the rotation of the Earth around the Sun is one year (about 365.25 days)
- The period of the rotation of the Moon around the Earth is close to one month (maybe add: and that's also how long it takes the Moon to pass through all its phases)

Let us try to be more precise with the Moon. We will fix the origin at the Earth (geocentric model) because then all the movements are circular (in the heliocentric model, the Moon makes a much more complicated trajectory).

Exercise: Assume that the Moon goes around the Earth in 27.32 days, and the Sun in 365.25 days (with respect to the fixed stars), and that they rotate in the same direction. Assume that at some moment the three bodies are aligned. Calculate when they will be aligned again.

Solution:
In t days the Moon covers an angle of 360 t/27.32º and the Sun covers an angle of 360 t/365.25º. Suppose we start with the three bodies aligned. Then the next alignment in the same order occurs when the Sun has covered a fraction of a turn and the Moon has covered a full turn plus the same fraction of a turn.



Calculating the period between alignments

This means that it occurs for t satisfying the equation

$$360\,\frac{t}{27.32} = 360 + 360\,\frac{t}{365.25}$$

whose solution is t = 29.53 days.

Note that to define what is “one turn”, we need to fix a reference frame. Thus we have different definitions:

- One *sidereal month* is 27.32 days, and it is the period that the Moon takes from being in the direction of any reference star to being in the same direction again. (It is also the period for the Moon between two passages in the vertical plane containing the slanted axis of the Earth, on the same side of the Earth.)
- One *synodic month* is 29.53 days, and it is the period that the Moon takes from being in the direction of the Sun to being in the same direction again.

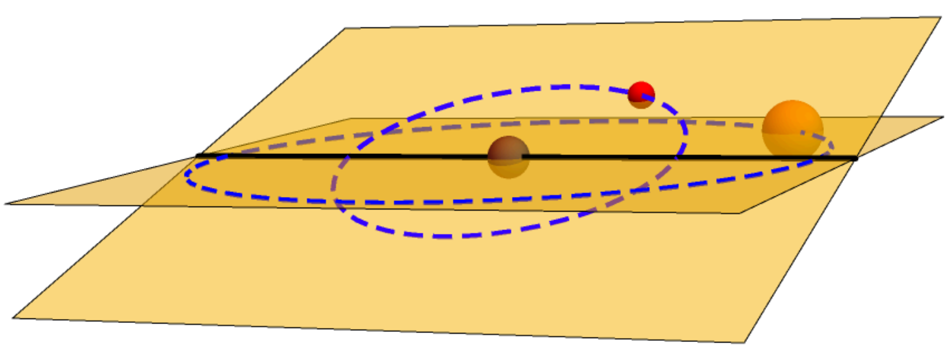
One synodic month is the period between two full moons. During this period the moon goes through all its phases.

Note that this also happens with the rotation of the Earth around itself

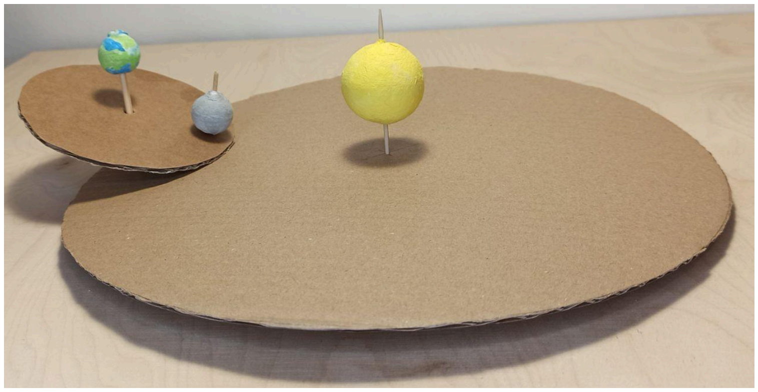
- One *sidereal day* is 0.9972 days (23 h 56 min 4.0905 s), and it is the period on which the skies appear to make one turn around the Earth (the Earth is in the same orientation with respect to the stars).
- One *(solar) day* is the period on which the Sun appears to make one turn around the Earth (the Earth is in the same orientation with respect to the Sun). It is also the period between two moments when the sun is at its zenith (highest point), the solar noon.

4. Tilt of the Lunar orbit plane

With this current model, it seems that every month, we would have a solar eclipse and a lunar eclipse. Why don’t we have eclipses every month?
The orbital plane of the Moon around the Earth is not the same as the orbital plane of the Earth around the Sun (called the *Ecliptic*).
In reality, the two planes make an angle of about 5º. You can exaggerate the angle in the model, since the relative sizes and distances are already off.



Task: Build and draw a heliocentric model taking into account the tilt of the Lunar orbital plane, and review the definitions you gave in step 3 with this new model.



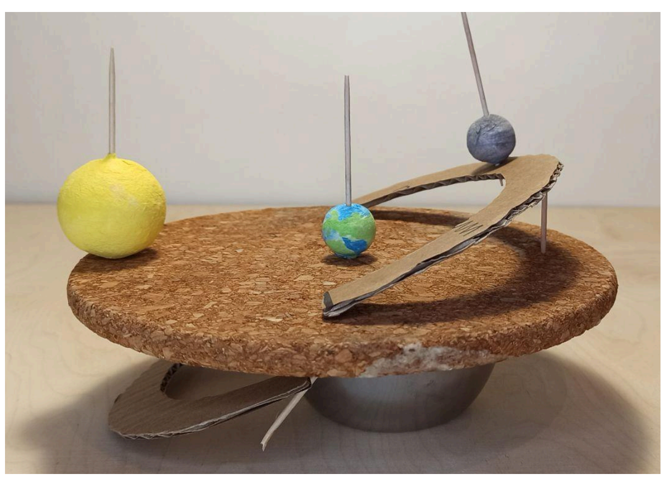
You can also see the interactive app Moon Phases and Eclipses.

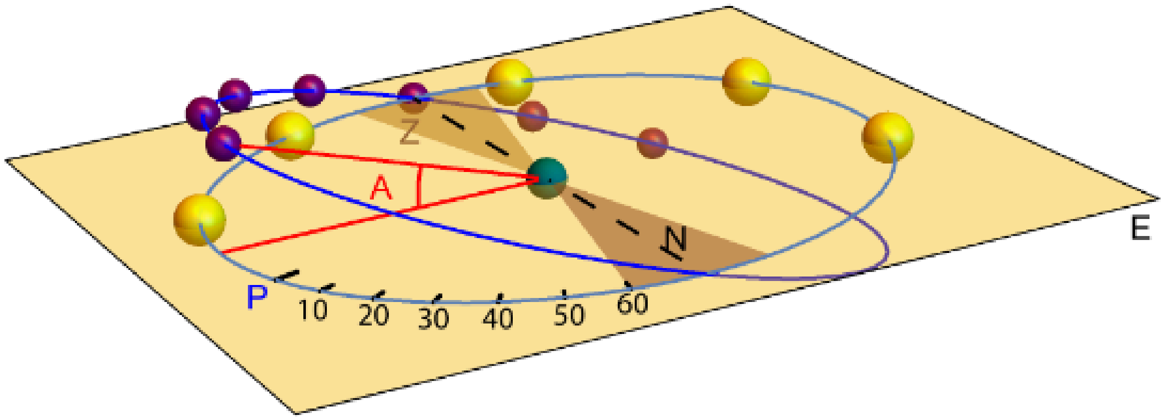
The lunar orbital plane and the ecliptic intersect in a line, called the *Nodal line*. The Sun, Earth and Moon can only be aligned on this line. The Sun is on the Nodal line roughly twice a year (two seasons for eclipses in a year). However, the eclipses are rarer because the Sun is not always in the nodal line when the Moon crosses that line.

Note: A subtlety not in our model: The Nodal line changes slowly (nodal precession), so the seasons for eclipses are not locked to year's seasons.

Task. Build and draw a Geocentric model considering the tilt of the lunar orbital plane.

This model is equivalent to the heliocentric (centered in the Sun) model, but it will give more insights and easier calculations. Put the Earth in the center of the cork plate. The Sun, the Moon, and all the stars lie (apparently) in a sphere centered on the Earth, the *sky dome*.





Geocentric model: The Sun (yellow) moves on the Ecliptic plane (E), and the Moon (purple) on its orbit plane at a 5.14º tilt angle (A). The Nodal line (N) is the intersection of both planes. For an eclipse to occur, both the Sun and the Moon must be within an 18.5º neighborhood of the nodal line (brown zone Z). The First Point of Aries (P) marks the origin of the ecliptic longitude.

The apparent path of the Sun is the Ecliptic. It is convenient to make the Ecliptic plane horizontal in our model. We can fix the Ecliptic's origin at the Sun's position on the first day of spring (in the Northern Hemisphere) in March; this point is called the *First Point of Aries*. The *Ecliptic longitude* is the angle measured from the First point of Aries along the Ecliptic. The Sun increases its Ecliptic longitude by approximately 1 degree per day, taking 365.25 days to complete its rotation of 360 degrees around the Earth.

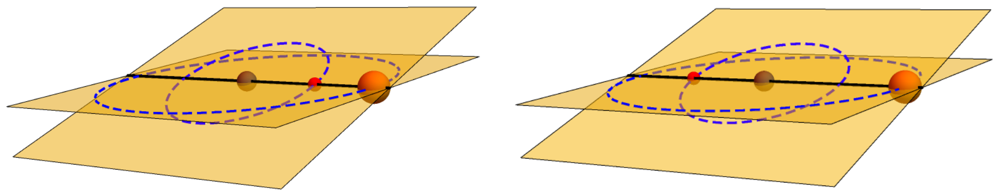
The Moon revolves around the Earth on a plane tilted 5.14º with respect to the Ecliptic, and it takes one sidereal month (27.32 days) to make one turn. The Lunar orbit plane crosses the Ecliptic in the Nodal Line.

Exercise: Identify again the phases of the Moon and the eclipses.

Note that because the angle between the two planes is small, it makes sense to project the Moon on the Ecliptic plane and measure its movement through its Ecliptic longitude.

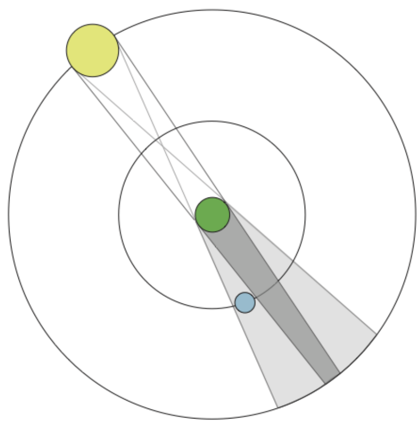
When the Sun and the Moon have the same Ecliptic longitude, it is a new Moon. After one synodic month (29.53 days), the Sun and the Moon have again the same Ecliptic longitude, and it is a new Moon again. When the difference between their Ecliptic longitudes is 180º, it is a full Moon, and when the difference is 90º, it is the first or last quarter Moon.

A solar eclipse is a new moon occurring on the nodal line, and a lunar eclipse is a full moon occurring on the nodal line. Since the Sun and the Moon have some width, there are some margins around the nodal line (about 18.5º) where an eclipse can occur.



Solar and Lunar eclipses

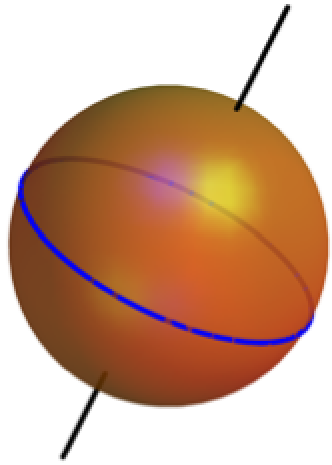
When the light of the Sun reaches the Earth, this creates a zone of umbra (shadow) where no light of the Sun penetrates, and a region of penumbra where part of the light of the Sun penetrates. It could occur that the Moon crosses the penumbra without crossing the umbra. In such a case we speak of penumbral lunar eclipse. Such an eclipse can be total or partial.



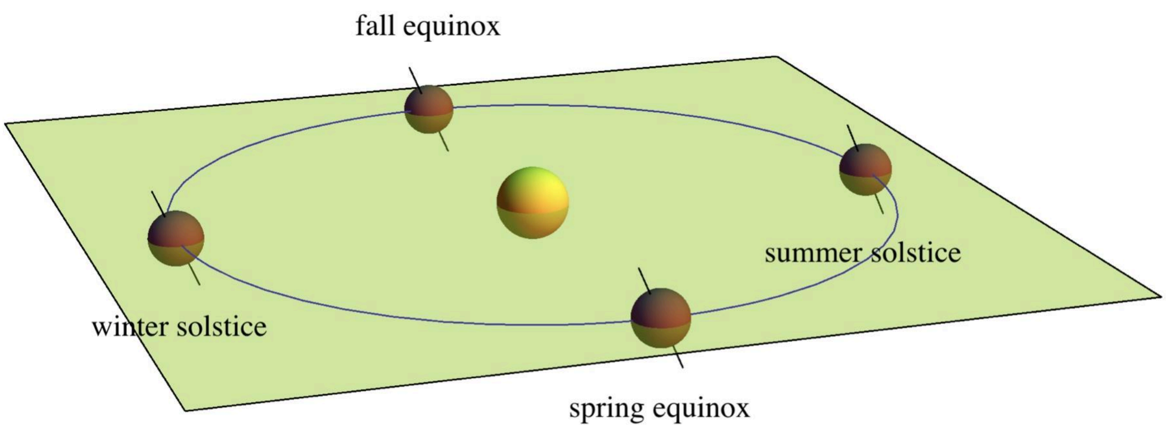
Penumbral total lunar eclipse

5. Extra activity: The seasons.

Introduce the tilting of the axis of rotation of the Earth to your model(s).



The axis of the Earth makes a fixed angle with the Ecliptic plane. This angle is responsible for the mechanism of the seasons.



Question: When do the first days of summer, autumn, and winter occur (note that the answer depends in which hemisphere you are located)? Discuss why the seasons (cold and warm) are caused by the tilt of the axis of the Earth (23.44º). Make your students deduce in which direction the Earth's axis is tilted.

Answer:
The First point of Aries is the point of the Ecliptic where the sun is on the first day of spring in the Northern hemisphere. It is the origin of the ecliptic longitude. Thus, spring is by definition located at ecliptic longitude 0º, and the summer, autumn and winter (in Northern hemisphere) are located at 90, 180º, and 270º respectively. In the Southern hemisphere, the first point of Aries corresponds to the first day of autumn, and the following points correspond to winter, spring, and summer, respectively.

If the North Pole is upwards in your model, it should be tilted 23.44º towards the Sun on the first day of summer. If you prefer the South Pole upwards, it should be tilted in the direction opposite to the Sun.

Compare the different angles with which the Sun reaches the Earth in summer and winter at a particular location on Earth (for instance, your location) and the long days and nights at the Poles.

Note: The First Point of Aries is, by definition, at the intersection of the ecliptic plane and the Earth's equator plane. This line is not fixed with respect to the stars, because the inclination of Earth's axis changes (the *precession of the equinoxes*), but its movement is extremely slow, about one turn in 26,000 years. However, enough time has passed since the Ancient named that point as being on the tip of the Aries constellation, and today that point is actually in the Pisces constellation.

This precession, among other things, means that the period of the Earth's orbit with respect to the stars (*sidereal* year, 365.256363004 days) is about 20 minutes longer than the same orbit with respect to the Sun (*tropical* year, 365.24219 days). The tropical year is the usual year, the basis for the calendar.

6. Extra material: Other lunar months, and frequency of eclipses.

There are different notions of lunar month:

- Sidereal month (from the stars): It is the period for the Moon to make a turn with respect to the fixed stars. It takes 27.321661 days.
- Synodic month: From Greek Synode, meeting. It is the period of the Moon to get again to the same Ecliptic longitude as the Sun. It is the period of the Moon phases. It lasts 29.53059 days.
- Draconic month (from the mythical dragon said to live in the two nodes, and eats the Sun or the Moon during eclipses): It is the period for the Moon to go from one node to the same node again. It is slightly different from the sidereal month, 27.212220 days.
- Anomalistic month: The Moon moves in an ellipse, not a circle. This makes the Moon appear a bit bigger when it is closer to the Earth and a bit smaller when it is further. This is the reason that sometimes a solar eclipse is total (the Moon covers completely the Sun) and sometimes it is annular (the Moon covers a circle almost concentric with the Sun, leaving a visible ring of light or the Sun). The axis of that ellipse is turning. The period taking the Moon to go from its perigee (closest point in the orbit) to it again is an anomalistic month, which lasts 27.554551 days

The ancient Babylonians and Greeks had a quite developed model of the skyes. They defined the so-called Saros cycle, based on an observational coincidence:
A Saros cycle is

- 223 synodic months (exactly, by definition)

- 242 draconic months (almost exactly)
- 239 anomalistic months (almost exactly)
- 6,585.321347 days
- 18.029 years (18 years, 11 days, and 8 hours)

If an eclipse occurs, then after a saros cycle, the Moon is in the same phase again (an integer number of synodic months), crossing the ecliptic again (an integer number of draconic months), and with the same apparent size (an integer number of anomalistic months), so a nearly identical eclipse occurs. Greeks had annotated tables with all the eclipses in one Saros cycle (recorded in about 18 years), from which they could extrapolate all eclipses and (their type) for centuries.

Since a Saros exceeds an integer number of days by one third, (about 8 hours), after a Saros the Earth has rotated one third of turn, so from the same position on Earth, the eclipse will be seen with 8 hours difference (hence in a region of Earth offset by 120º longitude). Every three Saros (54 years and 34 days), a nearly identical eclipse may be seen from the same place, at the same hour.

The Saros and the integer numbers of days and months fit the Greek ideal that “numbers govern everything” and that numbers are just natural numbers or fractions, the latter being ratios between natural numbers. Although the Greeks discovered irrational numbers, that posed a deep crisis in their philosophy, and it was much nicer to think of the skies governed by these special numbers.

7. Conclusions

In this activity we have built several models. A model, in the mathematical sense, is a simplified representation of reality that allows us to explore and understand a system.

The first model we made had all three bodies moving in the same plane. We refined that model by making the Moon orbit a different plane than the Sun. In the last part, we included the axial tilt of the Earth, and we mentioned several refinements that we can consider to make the model more accurate.

A model is based on observations from nature, and it proposes a mechanism “as if” nature follows it. In section 3, we calculated the synodic month from the sidereal month. However, we can measure both directly from nature: By observing the position of the Moon with respect to the stars, we measure the sidereal month. By observing the phases of the Moon (i.e. its position with respect to the Sun), we measure the synodic month. Do these observations match the calculated value? Yes! This validates the model since the Moon behaves *as if* it rotates around the Earth in a circle, at a certain constant speed. Except that… In reality, the observations do **not** match the calculations if we measure with sufficient accuracy! This reveals that the model is incomplete. Then we have to refine and say that actually, the orbit is not exactly a circle but an ellipse, and the speed around is not exactly constant but it's faster in the part of the ellipse that is closer to the Earth. Newton's laws provide a more refined model of the skies that describes the orbits and the forces between celestial bodies. However, Newton's laws cannot explain all astronomical phenomena, and at some point, we need to use Einstein's relativity theory. And so on. There are still phenomena awaiting for a good model. This doesn't mean simpler models are wrong or useless. They still provide a good overview, insights, and approximations of reality. More refined models are more accurate and give deeper insights.

Resources

- Moon phases and eclipses app: https://imaginary.github.io/moonphaseseclipses

Create and Share!
Share the participants' findings using the hashtags **#idm314eclipses** and **#idm314**.