The Art Gallery and the Fortress Problem

Participants:

Ages 10–12 and higher.
No previous mathematical knowledge is required.

Preparations:

Printed templates.
Pencils of four different colors (for example, red, green, blue, yellow).
Blank paper sheets to create your own art galleries.
Another possibility is to play the activity outside with colored chalk, either in the schoolyard or on the street.

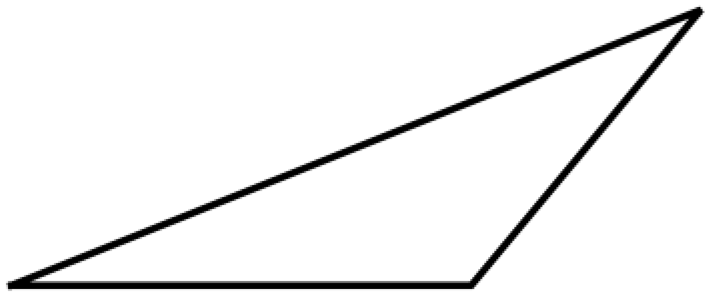
Activity 1. Protecting the art gallery with cameras

Imagine you're in an art gallery filled with amazing artwork. The gallery is not shaped as a rectangle or square as usual, but its floor plan has a fancy shape with lots of corners and twists. The shape is made up of straight lines that meet in corners (in mathematics it is called a *polygon*). Your mission is to strategically place cameras at specific spots in the gallery to make sure that every spot in the gallery can be seen and is under surveillance. But here's the catch: you have to use as few cameras as possible. And these cameras can only be placed at the corners of the gallery.

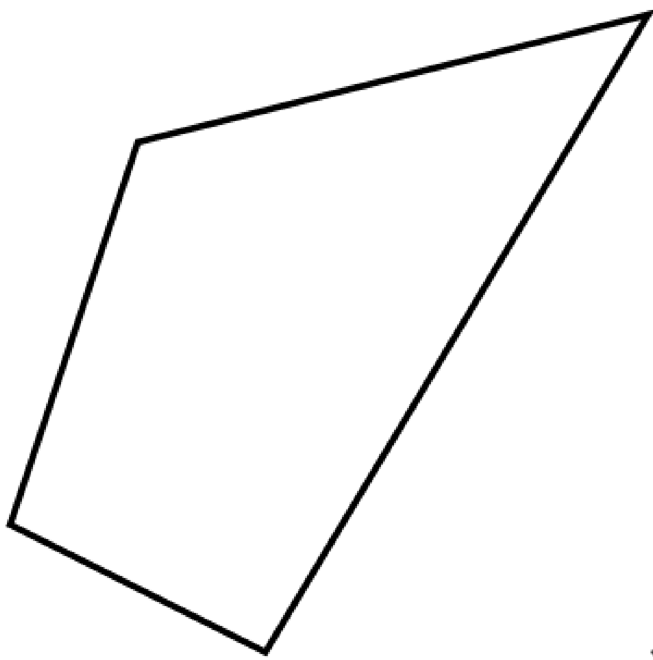
Your task is to place the fewest number of security cameras possible so that every single spot in the gallery is under surveillance. This is known as the "Art Gallery Problem."

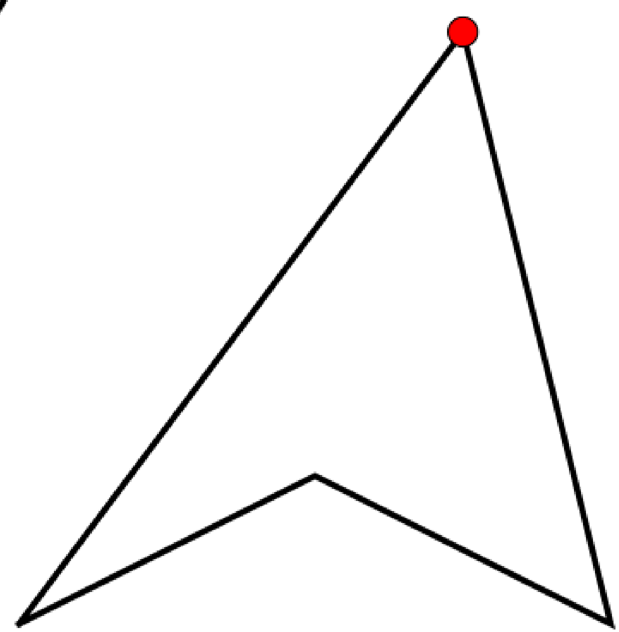
Using a pencil, you can draw straight lines starting from the camera to trace the area the camera can see. Remember that it cannot see through walls. You can also trace the camera's line of sight with a ruler to see what the camera can cover.

1. Let's start with exploring some simple cases:
   - For a gallery with a triangular layout, one camera is sufficient and can be placed at any corner (vertex), see below:

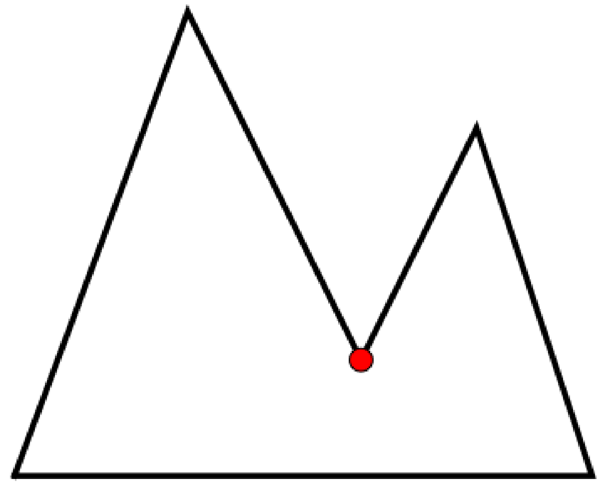


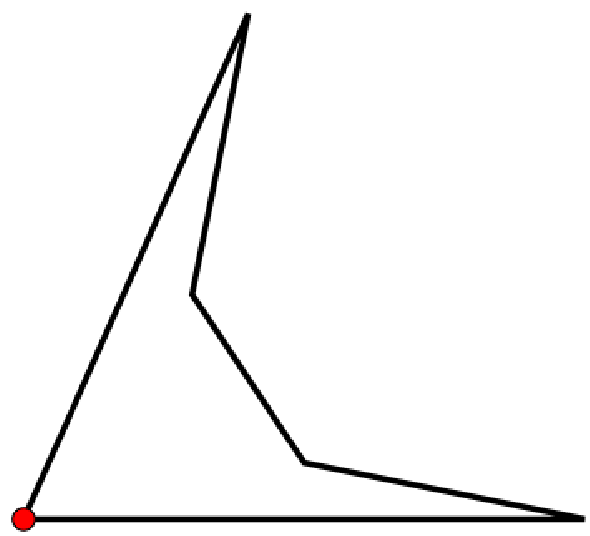
- In a four-sided gallery (called a *quadrilateral*), you also only need one camera. If it's a simple shape with corners pointing outwards (“convex”), one can place the camera at any corner, see left image below If the layout is more complex with some corners pointing inwards (“concave”), you have to choose the position of the camera more carefully to cover everything, see right image below. For the right gallery, a second position of the camera is possible. Can you find it?

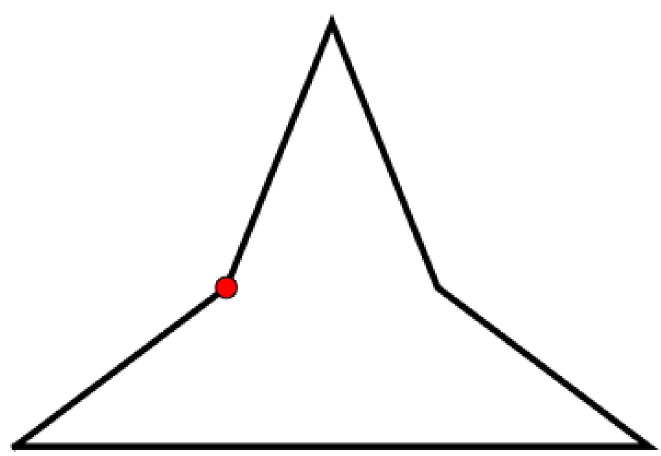




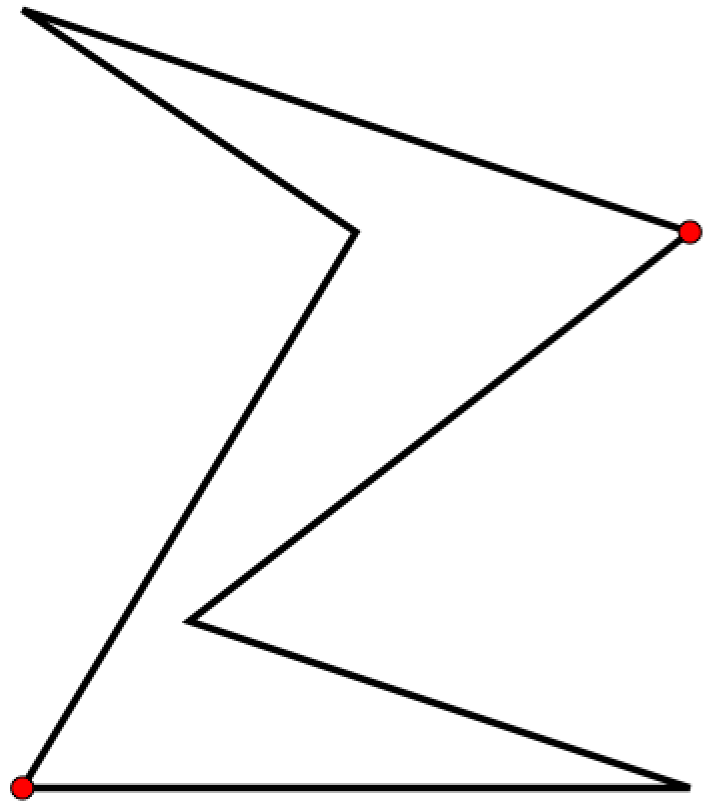
- A single camera is also sufficient in a gallery with five sides (a *pentagon* floorplan). It is always possible to place a unique camera in one corner, from which you can see the whole interior of the gallery. See the images below:



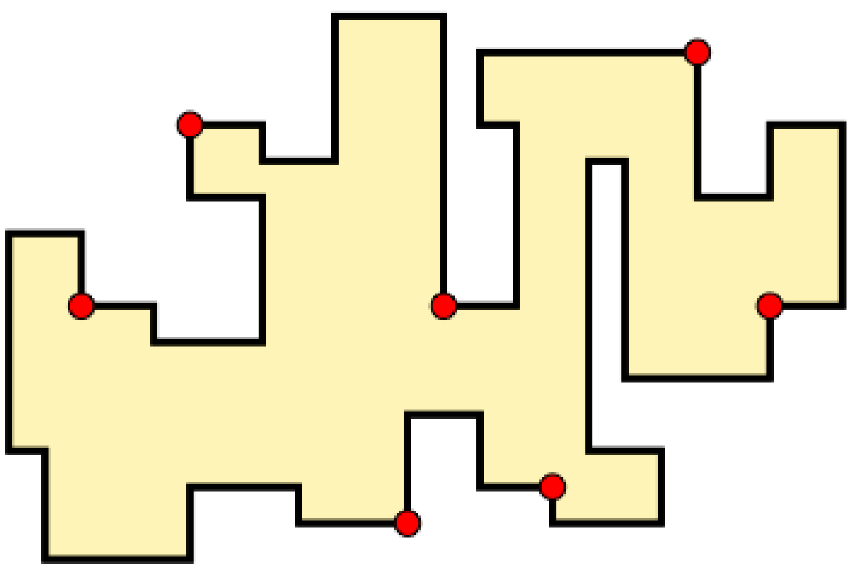




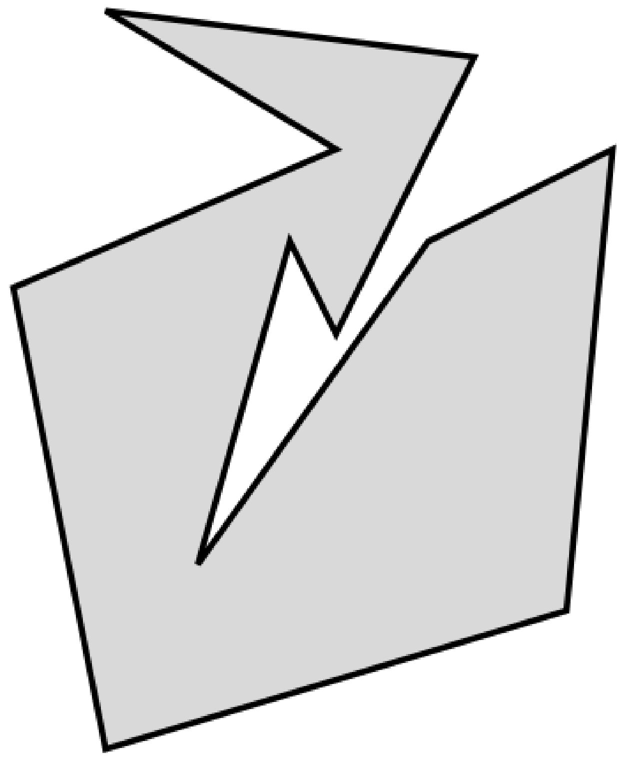
- However, two cameras are needed for the following gallery with six sides:



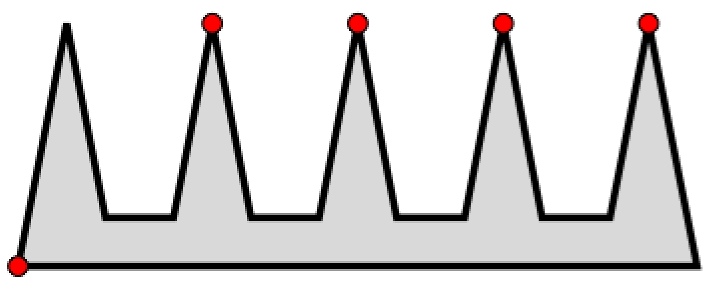
- You can check that seven cameras suffice for this gallery:



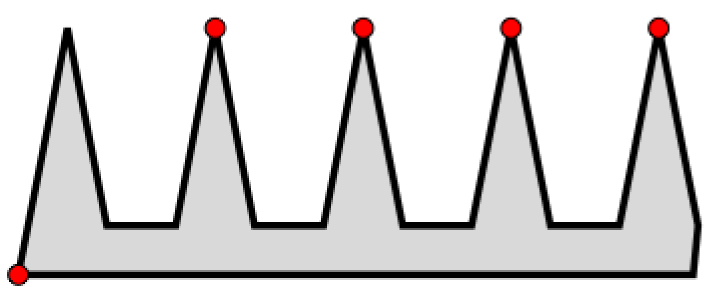
- Can you place only two cameras to observe the following gallery?

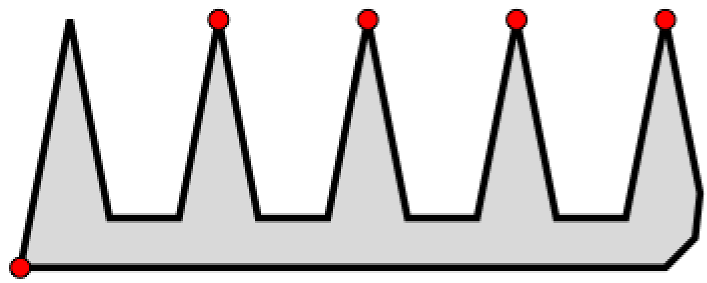


2. We now move to a mathematical rule. In the art gallery below with 15 sides, you need at least 5 cameras to cover it completely.



You can't use fewer than 5 cameras and still watch everything. The same is true for these two galleries with 16 and 17 sides.

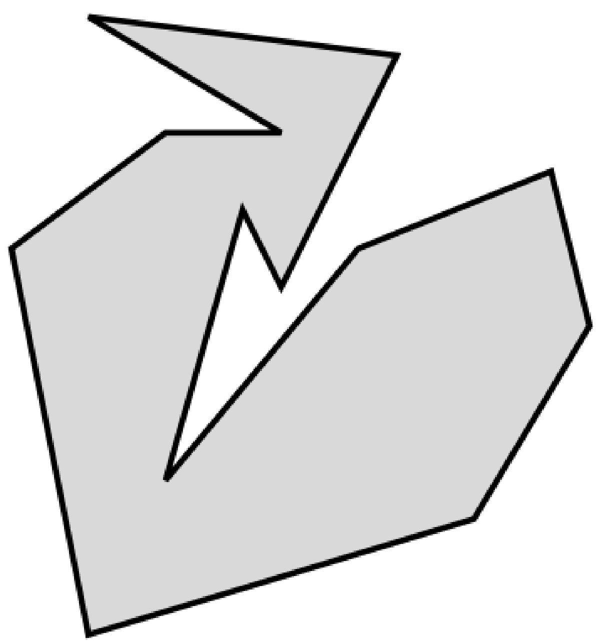




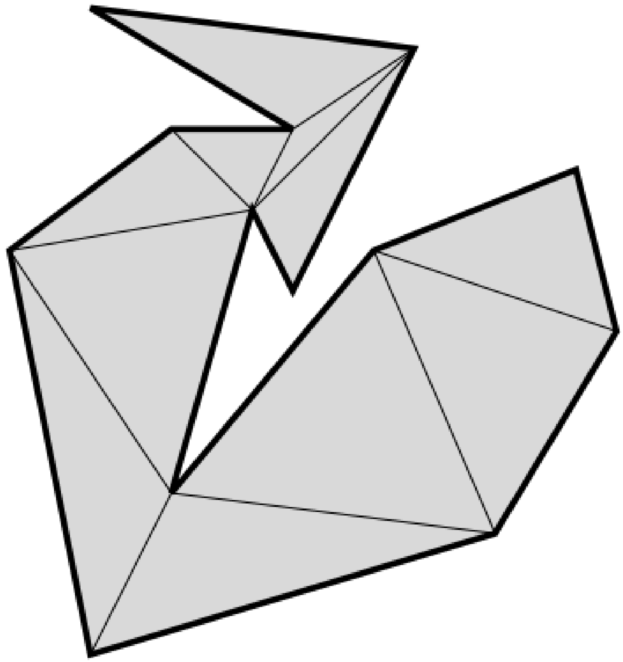
Note that 5 is the quotient of the division of 15, or 16, or 17 by 3. Revisit all the above examples and check that, in each case, you can watch the whole gallery with a number of cameras at most equal to the quotient of the division of the number of sides by 3.

The mathematician Václav Chvátal proved in 1975 that a number of cameras equal to the quotient of the division of the number of sides by 3 is sufficient for any gallery! As an example: for 6-sided galleries it would be 2 cameras, for 10-sided galleries it would be 3 cameras, for 23-sided galleries it would be 7. Interestingly, Chvátal's rule still works if you place the cameras inside the shape instead of just on the corners. So, it's a helpful guideline for setting up surveillance cameras in places with different and complicated shapes.

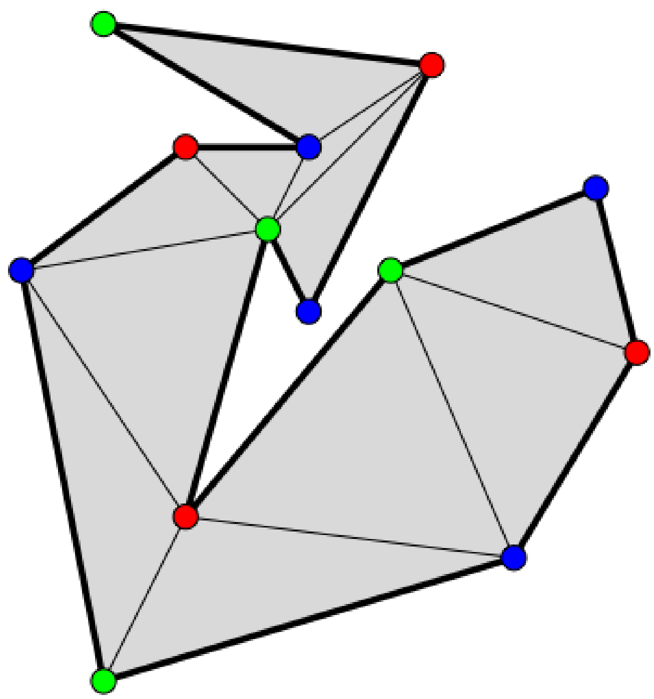
3. A very elegant proof simpler than that of Václav Chvátal was proposed by the mathematician Steve Fisk in 1978. It provides an **algorithm**, or a step-by-step plan, for where to place cameras. Let's take a closer look at how this algorithm works on the following gallery



- The first step is to divide our art gallery into triangles. The corners of the triangles are at the same locations as the corners of the original art gallery:

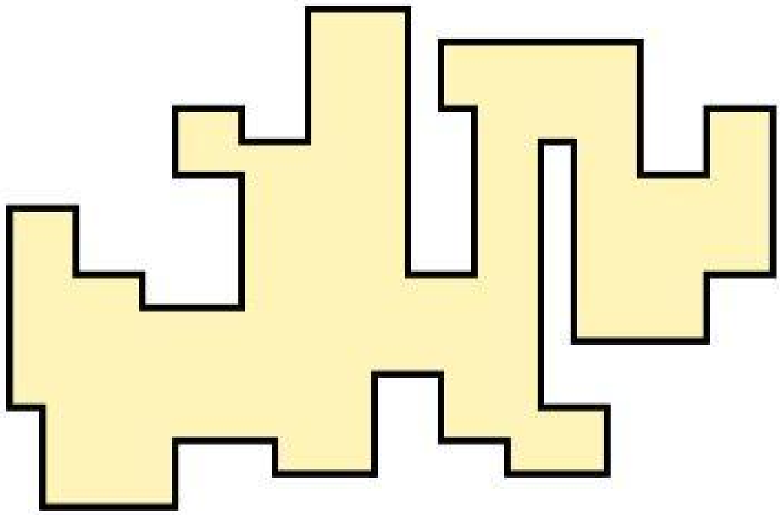


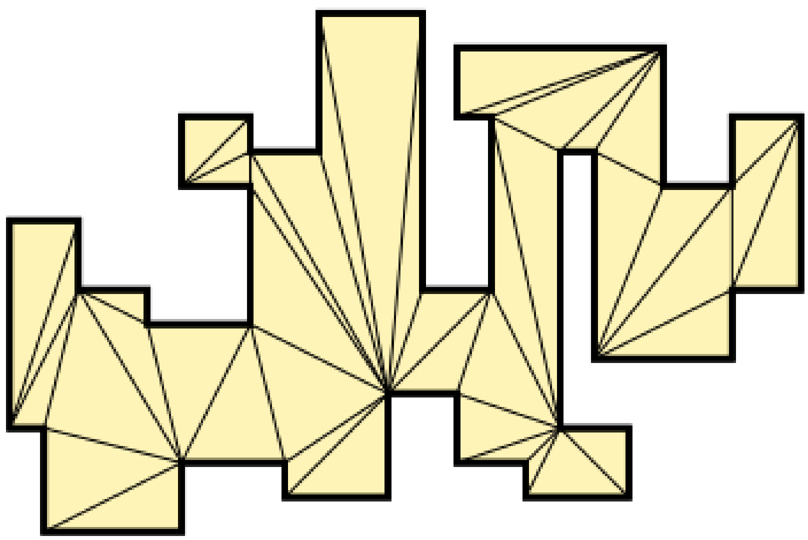
- The next step is to assign one of three colors (let's say red, green, and blue) to every corner so that each triangle ends up having corners of three different colors (this is always possible).

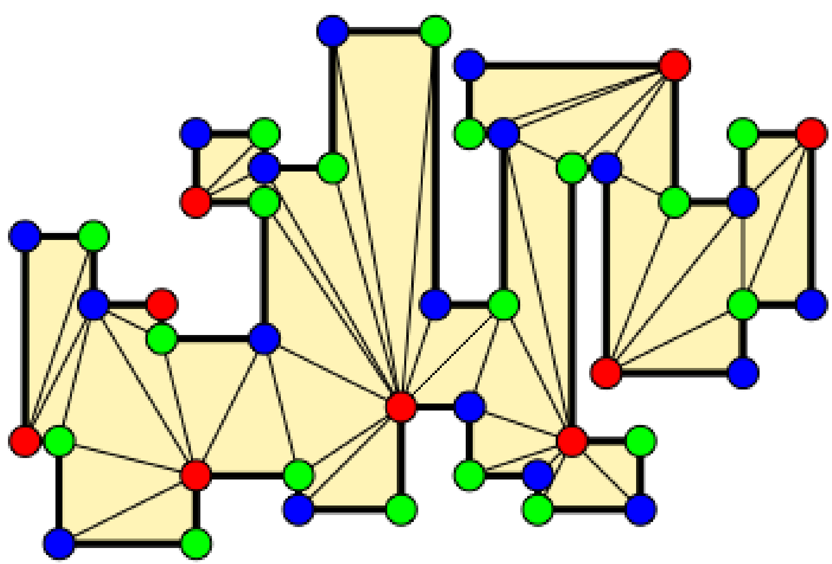


- Pick the color that shows up the least. In this case, there are 4 dots with the color red, 4 with green, and 5 with blue. So, we have two choices. We can choose the 4 red dots and solve the problem by placing the cameras at these red dots. We could also have placed the cameras at the four green dots. In both cases, the four cameras suffice to watch everything.

- Here is a more complicated example.

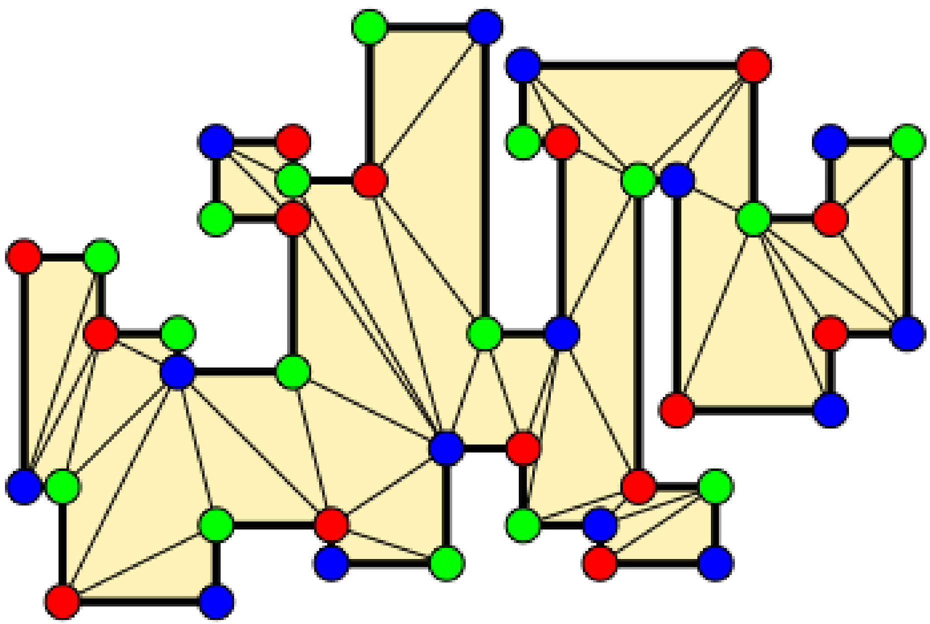




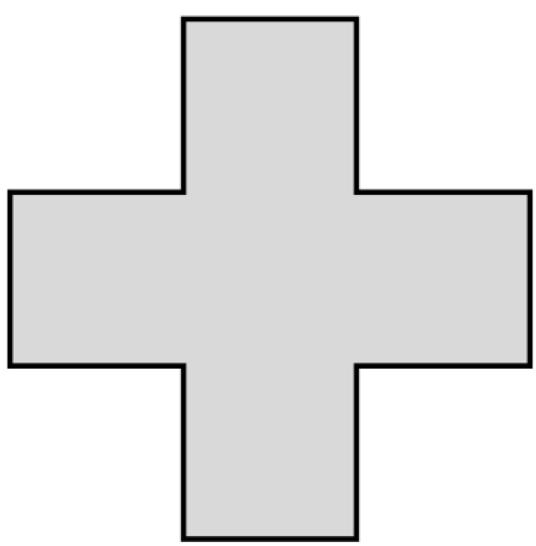


In this case, we have 9 red dots, 18 blue dots, and 19 green dots. Thus, by placing the cameras at the 9 red corners, we solve this example.

There is another interesting fact about this algorithm: Below, you find a picture of the same gallery but with different triangles than in the picture before. One says that “the *triangulation* is not unique”, meaning that there are different possibilities on how one can divide a gallery into triangles. With different triangles, we also have different colors of the dots at each corner of the triangle. This means that for one gallery, there is more than one solution on where to place the cameras. In the new triangulation below, there are 15 red dots, 15 blue dots, and 16 green dots. Hence, 15 cameras are needed, which can be placed either at the red dots or at the blue dots. This solution is not as economical as the one before. Fisk’s algorithm provides solutions, but they may not be optimal.

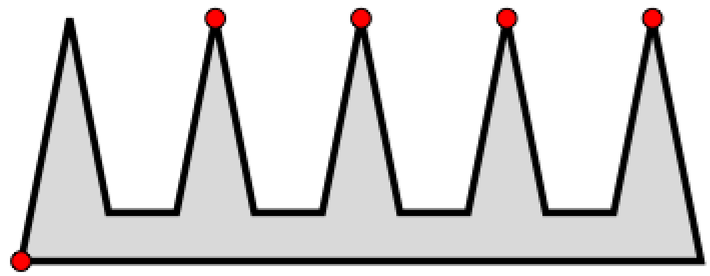


- Find different ways to divide this polygon into triangles and explore the placement of cameras in each case.



There are much better solutions than the ones provided by the algorithm. These solutions have only one camera. Can you find them?

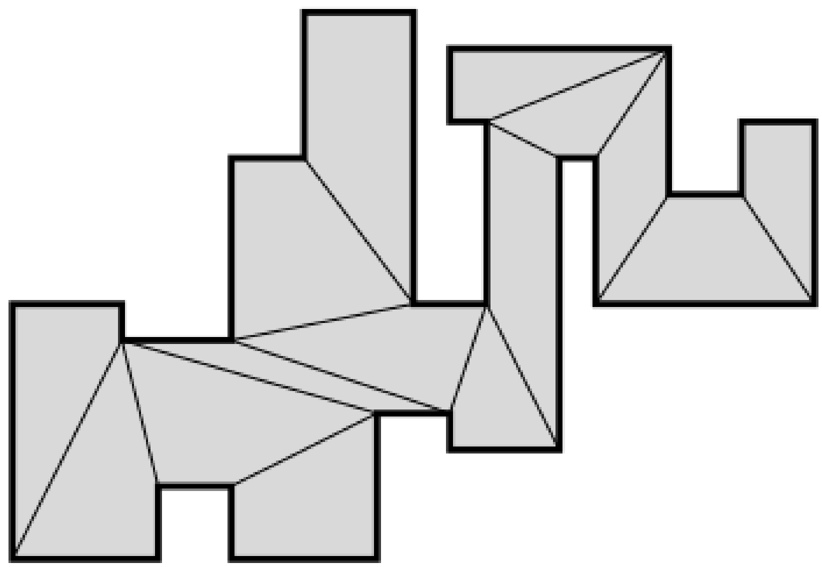
- Draw triangles for which the algorithm provides the solution below (some of the triangles are very thin):



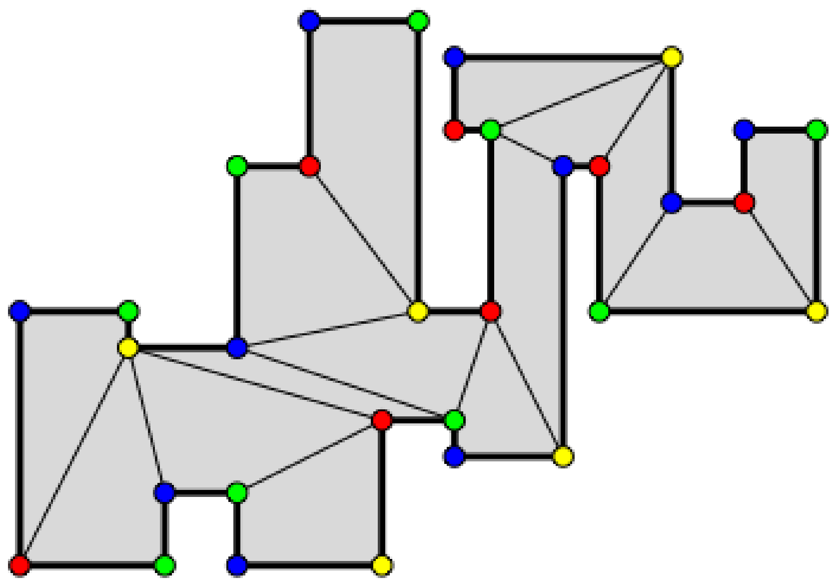
4. Draw other art galleries and explore the algorithm on them.

5. In 1980, Jeff Khan, Maria Margaret Klawe, and Daniel J. Kleitman found a more economical rule for a special kind of shape, a gallery with only right angles. Such a gallery is called an *orthogonal gallery*. As above, you can use a certain number of cameras to watch the entire area inside it. This number of cameras needed is different than in the case of the more general art gallery shape. Here, you do not divide the number of sides by 3 (as above), but by 4 – and again cut off the decimals. So, if your gallery has 20 sides, you are sure that at most 5 cameras suffice. If your gallery has 8 sides, at most two cameras suffice. Since one-fourth is smaller than one-third, this means that the art galleries with only right angles, in general, need *fewer* cameras!

   The basic idea here is similar to what we discussed earlier. We again want to divide our art gallery into smaller shapes. But since our art gallery has the special property of only having right angles, it is now possible to divide it into four-sided shapes called "*quadrilaterals*". The quadrilaterals have to be "convex". This means four-sided shapes that have straight sides and angles that are not bent inwards. (This was not possible in the case of the more general art gallery!)
   These four-sided shapes have their corners at the corners of the art gallery. This method isn't easy to do or prove, but it shows that you can efficiently cover the whole interior of this kind of art gallery with a specific number of cameras.

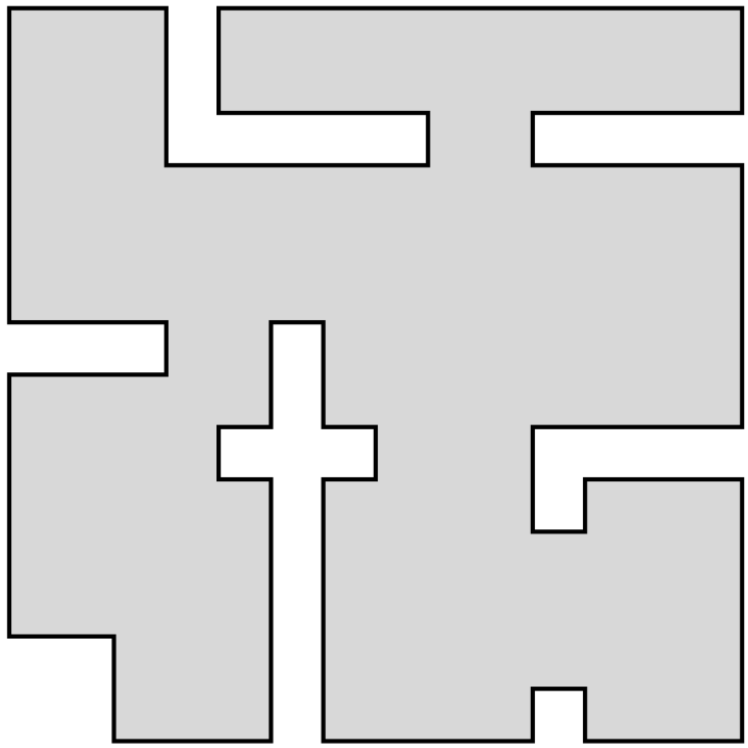


You can use four colors to paint the corners of these four-sided shapes in a way that each corner gets a different color, making sure that all four corners of each shape have all four colors used.



There are 6 yellow corners, 7 red corners, 9 green corners, and 10 blue corners. Remember that a camera placed at any corner of a convex quadrilateral can guard the full quadrilateral. We simply place the cameras at the yellow corners.
Since we now use four different colors instead of three, this then gives us a smaller number of needed cameras to watch every spot.

Execute the algorithm on this gallery:

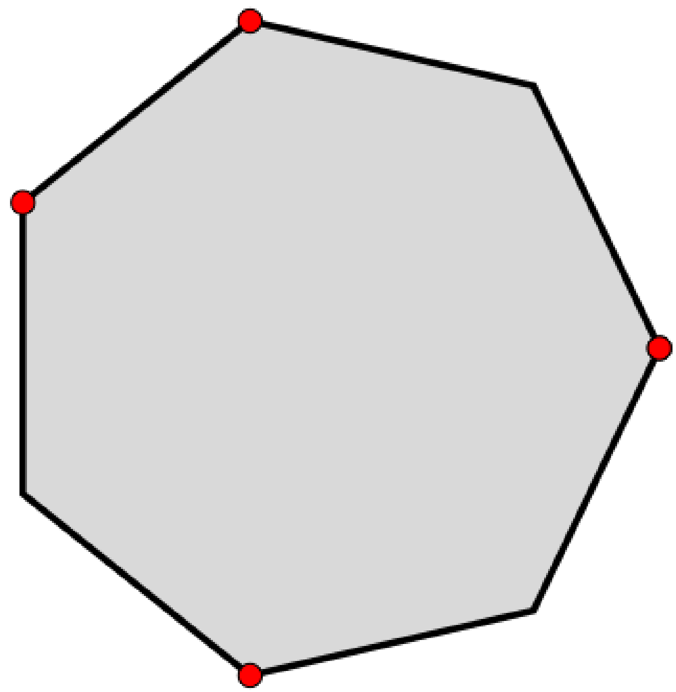


Draw other orthogonal art galleries and explore them.

Activity 2. The Fortress Problem

Imagine there's a fortress, and it's shaped like a closed figure with straight sides and corners. The challenge is figuring out how to place the fewest cameras (or guards) inside the fortress so that no matter where you are **outside** the fortress, at least one camera can see that spot. Just like in the Art Gallery Problem, we're only allowed to put cameras at the corners of the shape.

Given all fortresses with the same number of sides, we can find a number of cameras that suffice for all these fortresses. This number of cameras is the number of sides divided by 2, and the result is rounded up to the next whole number. So, for a 7-sided shape, you'd need at most 4 cameras (because half of 7 is 3.5, and we rounded up to the next whole number 4).



Draw other fortresses and explore them.

Mathematical background and resources:

The fancy shapes of the art gallery we described above can be mathematically described as "*polygons*". A polygon is a simple, closed shape made by connecting straight lines. "Simple" means the lines don't cross each other, and "closed" means the lines connect to form a complete shape with no gaps. For example, a triangle is a polygon with three straight sides, and a square is a polygon with four straight sides, while a cross is not a polygon. The sides meet at points, which are the corners of the shape. These corners are also called vertices.

Triangles, which are the simplest polygons, can be thought of as the basic building blocks of polygons. One can build every form of polygon by combining different triangles. Therefore, it is also possible to go the other way round and find the triangles that a polygon is made of. This process of dividing a polygon into triangles is called "*triangulation*".

If you want to know more about this, here is a book recommendation: *Art Gallery Theorem and Algorithms*, by Joseph O'Rourke, Oxford, University Press, 1987. The book can be consulted or downloaded here.

Create and Share!
Share the participant's galleries and fortresses you created using the hashtags **#idm314gallery** and **#idm314**.