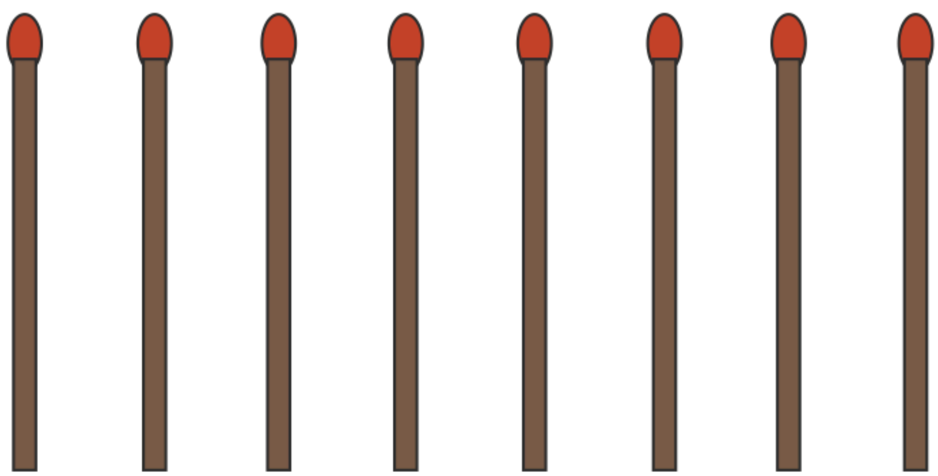
INTERNATIONAL DAY OF MATHEMATICS MARCH 14

# Math Games

**Participants:**
Ages 10 and up. Some activities are suitable for ages 15 and up.

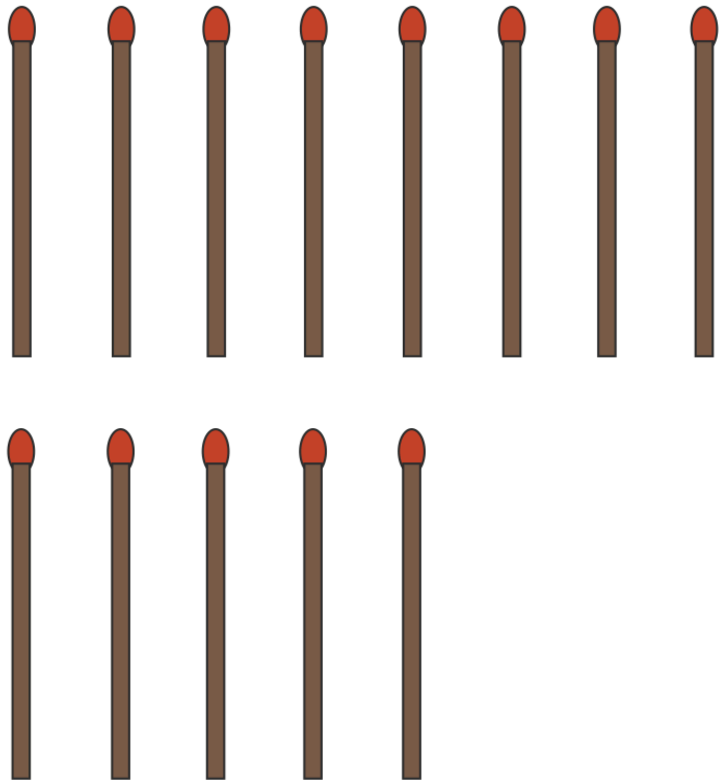
**Exploring the game of Nim:**
This is an ancient strategy game for two players whose origin is uncertain (some believe it is from China). Many variants exist.

1. *First variant:* any number of sticks are aligned in a row (you can use matches, pencils, or fingers).

   Each player can remove 1, 2, or 3 sticks at their turn. The winner is the one who takes the last stick: this is the *greedy game*. Experiment with different strategies. Show that if the number $n$ of sticks is not divisible by 4 when it is one player's turn, then this player has a winning strategy: indeed, the remainder of the division by 4 is 1, 2, or 3. The strategy is to take a number of sticks equal to this remainder.
2. *Second variant:* any number of sticks is aligned in a row. Each player can remove 1, 2, or 3 sticks at their turn. The loser is the one who takes the last stick: this is the *misère game*. Show that if the remainder of the division of the number $n$ of sticks by 4 is not equal to 1 when it is one player's turn, then this player has a winning strategy. Explain the strategy.

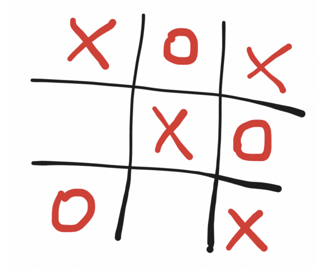
3. One possibility is to have two players play together, one of whom has explored the greedy game and the other the misère game: several games are played, alternating the version played, the person who starts, and changing the number of sticks.
4. *Third variant:* sticks are aligned in several rows, and each row can contain any number of sticks.



The two players alternate, taking any number of sticks from any row. The winner is the one who takes the last stick.

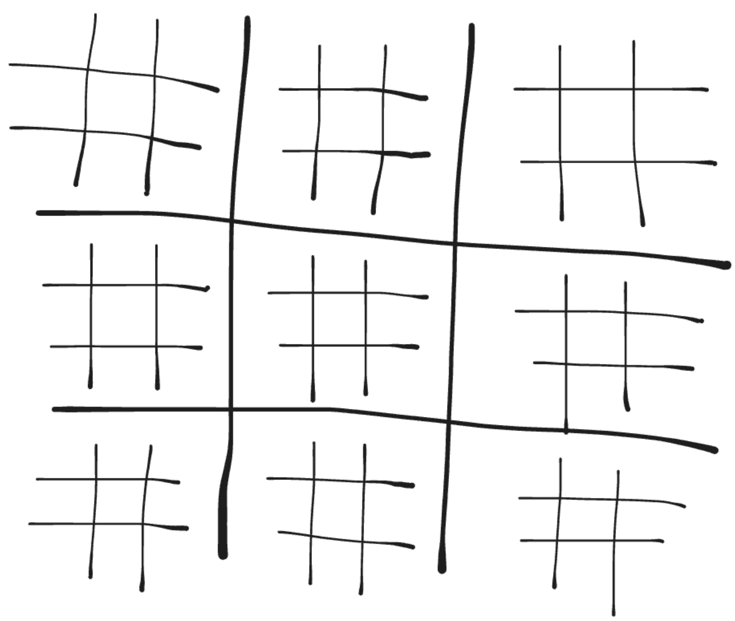
   a. Experiment with 2 rows, and show that if a player can make a move bringing the rows to the same number of sticks: (1, 1), (2,2), (3, 3), (4,4), etc., then that player has a winning strategy.
   b. If we take again 2 rows and change the rule so that the loser is the one who takes the last stick, show that if a player can make a move bringing the rows to the same number of sticks greater than 1: (2,2), (3, 3), (4,4), etc. then that player has a winning strategy.
   c. Experiment with 3 rows and check that a move to one of the following configurations is a winning strategy: (1, 2, 3), (1, 4, 5), (1, 6, 7), (1, 8, 9), (2, 4, 6), (2, 5, 7), (3, 4, 7), (3, 5, 6), (4, 8, 12), (4, 9, 13), (5, 8, 13), (5, 9, 12). (The idea is that by two moves, you can jump to a simpler winning configuration, either coming before in the list or with fewer rows as in a.)
   d. Experiment with 4 rows and check that a move to one of the following configuration is a winning strategy: (1, 1, $n$, $n$) for any $n$, (1, 2, 4, 7), (1, 2, 5, 6), (1, 3, 4, 6), (1, 3, 5, 7), (2, 3, 4, 5), (2, 3, 6, 7), (2, 3, 8, 9), (4, 5, 6, 7), (4, 5, 8, 9), ($m$, $m$, $n$, $n$) for any $m$, $n$.
   e. The general case is for ages 15-18. It requires using numbering in base 2. It is explained below in Appendix 1.

**The game of Tic-Tac-Toe and some generalizations:**
The Tic-Tac-Toe game can be traced back to ancient Egypt. It is played on a 3x3 grid. Player A writes an X in one square. Player B writes an O in another square. Players A and B alternate in writing X's and O's in the empty squares. The winner is the first player who succeeds in aligning three of their marks, either horizontally, vertically or diagonally.

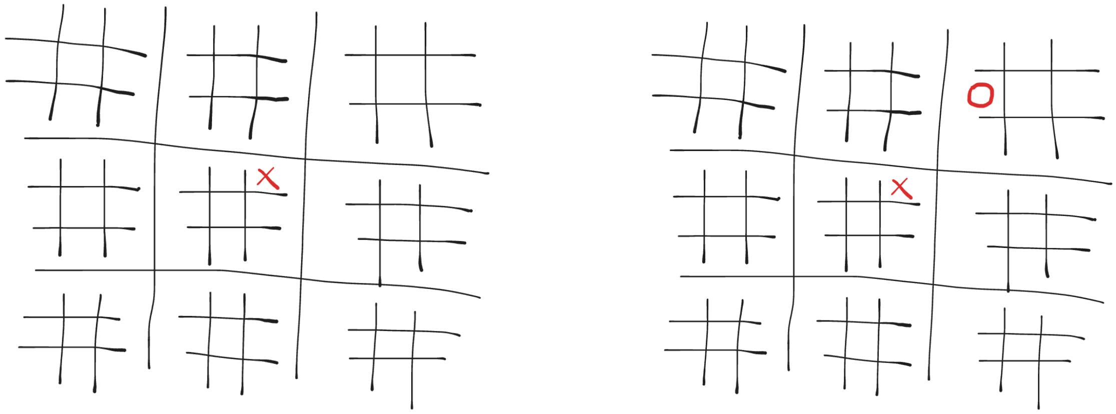



- Show that if all players play cleverly, then no one wins. This can be done by determining a strategic move for every possible move of the opponent.
- One generalization is to play the game on a 3x3x3 grid. Experiment that Player A will surely win when placing an X at the center on the first move. Determine her/his strategy, depending on the moves of Player B.
- Tic-Tac-Toe can also be played on a 4x4 grid. The winner is the first player that either places 4 of their marks in a horizontal row or in a vertical row, or in a 2x2 square, or at the four corners of the grid. Experiment strategies for that game.

**The game of Ultimate Tic-tac-Toe:**
In a big 3x3 grid, each square contains a smaller 3x3 grid to play Tic-Tac-Toe.



The first player can play with their symbol (X or O) in any small grid. Each move determines which of the small grids will be used in the next turn. For instance, if the first player places an X in the upper right square of the central small grid, the next player must place an O somewhere in the small grid located in the upper right corner of the big grid.

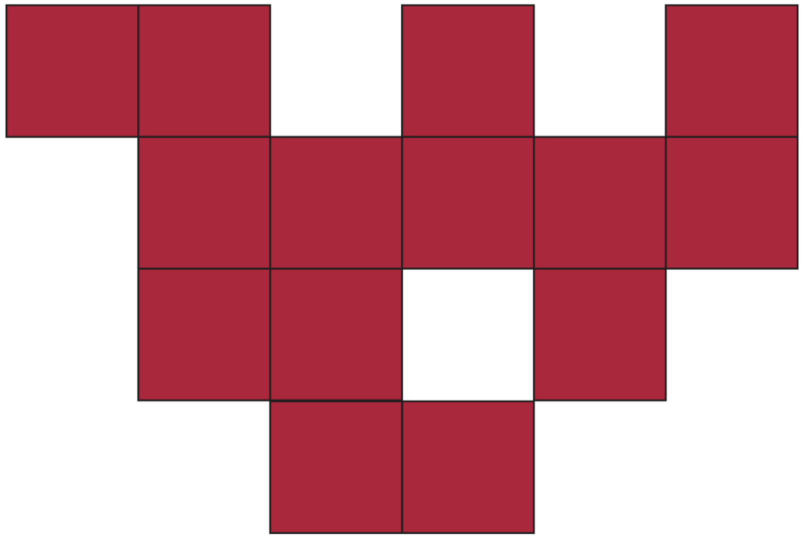


*Example of the first three moves. For the first move, player 1 can put an X anywhere. Since player 1 played in the upper right corner, then player 2 would have to put an O somewhere in the upper right grid. For the third move, player 1 has to put an X in the small grid on the left of the second row.*

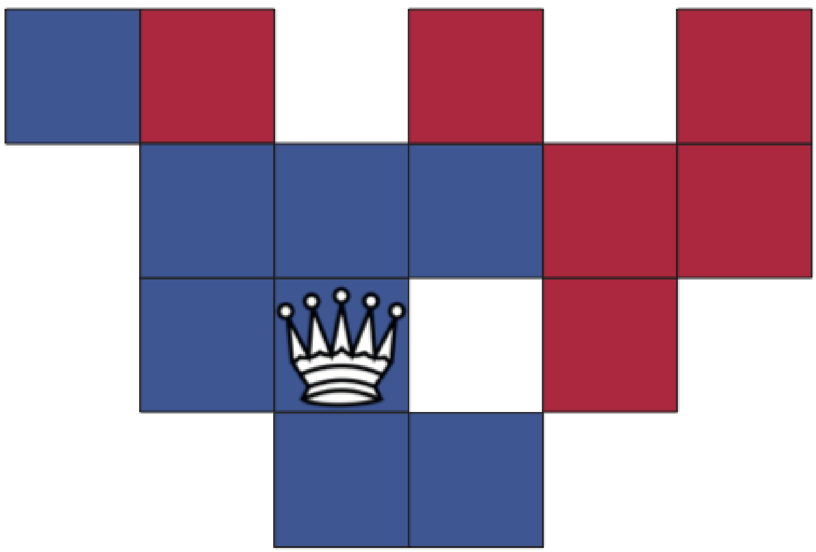
Each time a player manages to make three in a line in a small grid, that big square of the big grid belongs to them. The winner is the player who manages to win the big Tic-Tac-Toe grid by winning three smaller Tic-Tac-Toes in line.
If a player is sent to a grid that is full, or that has already been won by anyone, then that player can choose any other small grid to place the next X or O.

**Queens on chessboards**
On a chessboard, queens can move horizontally, vertically and diagonally. A queen is threatened by another queen if the second queen can move to the position of the first queen, either horizontally or vertically or diagonally.

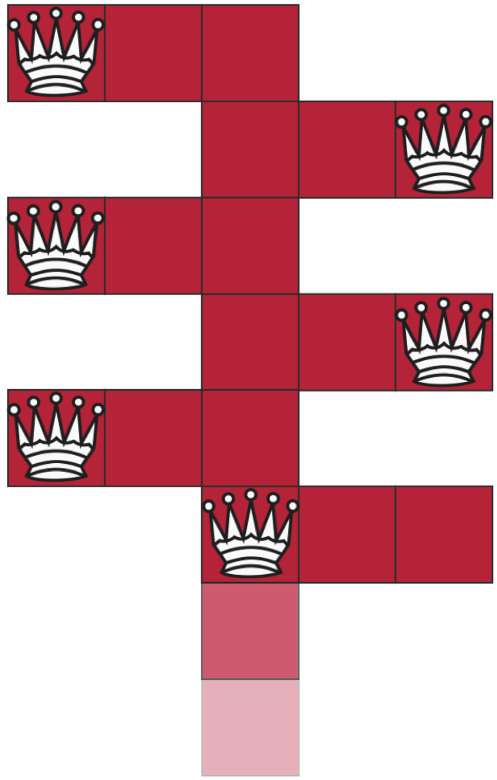
- Take a 4x4 chessboard and place four queens, so that no two queens threaten each other.
- Same question with a 5x5 chessboard and five queens.
- Same question with a 6x6 chessboard and six queens.
- It is possible to generalize the shape of the chessboard to a polyomino, which is a connected set of squares, each square being adjacent to at least another square by a side.



The question is then to determine the minimal number of queens necessary to threaten all squares of the polyomino. Here are the squares threatened by one queen.



- For this polyomino, three queens suffice. Can you place them?
- The general result is that if the polyomino has $n$ squares, then it will always be possible to threaten all squares with at most $m$ queens, where $m$ is the integer part of $\frac{n}{3}$. In the following example which has 18 squares (or 19 or 20 if we add one of the transparent bottom squares), it is not possible to use less than 6 queens to threaten all squares.



This can be generalized to a polyomino with $3n$ squares (or $3n + 1$, or $3n + 2$ squares) for which at least $n$ queens are necessary to threaten all squares.

- Draw other polyominoes and determine in each case the minimum number of queens necessary to threaten all squares.
- Proving that at most $m$ queens suffice, where $m$ is the integer part of $\frac{n}{3}$ is for ages 15-18. It is explained below in Appendix 2.

**Create and Share!**
Create new rules for the above-mentioned games and share them using the hashtag **#idm314games** and **#idm314**.

**References**

For Ultimate Tic-tac-Toe:
https://en.wikipedia.org/wiki/Ultimate_tic-tac-toe
Online game with AI: https://www.uttt.ai/

For the queens on the polyominos (in French):
https://accromath.uqam.ca/2022/09/des-dames-sur-detranges-echiquiers/
Online game: www.erikaroldan.net/queensrooksdomination

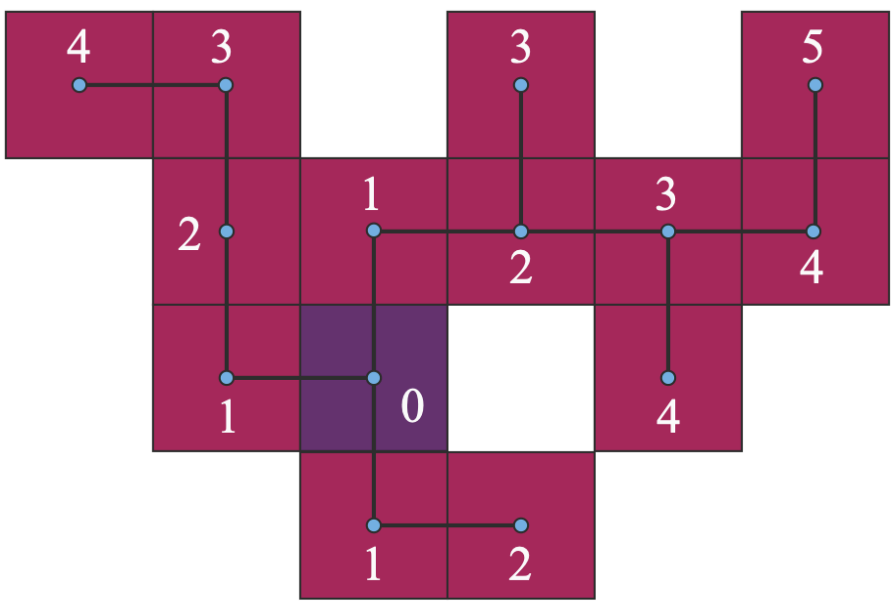
**Appendix 1. The general case for the game of Nim:**
Consider a game of Nim with $m$ rows, containing respectively $n_1$, ...., $n_m$ sticks. Write each of the numbers $n_1$, ...., $n_m$ in base 2. Each number is then a sum of powers $2^i$ for some $i = 0, ...., N$. Count how many times a given power $2^i$ appears in all $n_1$, ...., $n_m$. If it appears an even number of times, then let $a_i = 0$. If it appears an odd number of times, then let $a_i = 1$. Here is an example. We have three rows with respectively 9, 5 and 6 sticks. Hence $n_1 = 9 = 2^3 + 2^0$, $n_2 = 5 = 2^2 + 2^0$, $n_3 = 6 = 2^2 + 2^1$. Then $a_3 = 1$, $a_2 = 0$, $a_1 = 1$, $a_0 = 0$.

**Claim:** a winning strategy for a player is to make a move bringing all $a_i$ to 0.

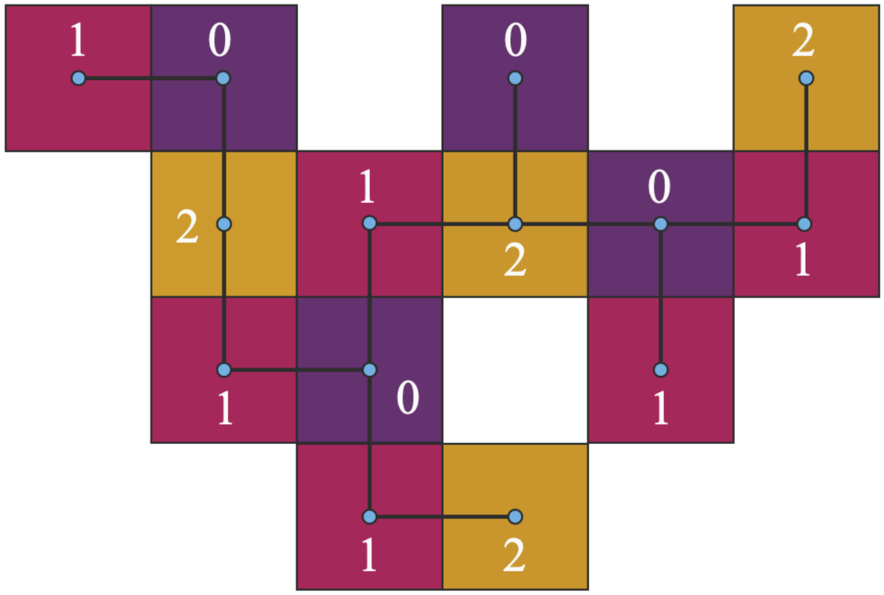
1. Show that if all $a_i = 0$, $i = 0, ...., N$, then any move will bring at least one $a_i$ to be nonzero.
2. Show that if some $a_i$ are nonzero, then there exists a way to remove some sticks in one row so as to bring all $a_i$ to 0.

It is recommended to experiment with several examples before trying to prove the general rule.

**Appendix 2. An algorithm to place the queens on a polyomino:**
Let's take a polyomino and mark the center of the squares. We can assume that the squares have side lengths equal to 1. We decide to mark a central square and to measure the minimal distance from its center to the center of the other squares when moving horizontally or vertically from center to center.




Then we compute the remainder of the division of this distance by 3. The remainders are 0, 1 or 2. All squares with the same remainder of the distance are colored with one color and squares with different remainders are colored with different colors.




1. Explain why placing queens on all squares of a given color allows threatening all the squares of the polyomino.
2. A minimal solution is then obtained by taking squares of one color appearing the least often. Identify the two solutions in the example. Explain why this color appears at most $m$ times, where $m$ is the integer part of $\frac{n}{3}$.
3. Explore the algorithm on other polyominoes.
4. Draw polyominoes for which we can use fewer queens than the number given by the algorithm.